Phase unwrapping using a regular mesh grid

L.R. Berriel-Valdos[1], C. Olvera-Olvera[2], J. G. Arceo Olague[2], E. de la Rosa M.[2], E. Gonzalez-Ramirez[2], T. Saucedo-Anaya[3], J. Villa-Hernandez[2], I. de la Rosa[2]

[1] *Departamento de Óptica, Instituto Nacional de Astrofísica, Óptica y Electrónica (INAOE). Luis Enrique Erro #1. Santa María Tonantzintla.*

[2] *Unidad Académica de Ingeniería Eléctrica, Universidad Autónoma de Zacatecas, Lopez Velarde 801, Zacatecas, Zac.*

[3] *Unidad Académica de Física, Universidad Autónoma de Zacatecas, Av. Solidaridad Esq. con Paseo la Bufa S/N, Zacatecas, Zac.*

*e-mail: `berval@inaoep.mx`

Abstract

Phase unwrapping is a key step in the fringe pattern analysis. Although there are many algorithms for recovering continuous phase from wrapped phase maps, many of them are computationaly heavy, even for smooth phase maps. The smoothness characteristics of a phase map allow the use of radial basis functions to model the unwrapped phase. This method helps to reduce the processing time when unwrapping the phase. The processing time can be reduced even more when the reconstruction does not take into account all the pixels of the phase map image. In this paper we describe an algorithm for phase unwrapping where the phase map is reconstructed from a subset of pixels of the phase image using radial basis functions (RBFs). The proposed method is compared with the algorithm based on the same radial basis functions (RBFs) but using all the phase image pixels.

1. Introduction

Interferometric methods are widely used to measure physical magnitudes such as deformation, stress, temperature, etc. [1, 2] in a non destructive and non invasive way. These magnitudes modulate a fringes pattern called interferogram which contains the information about the related physical magnitude. Demodulation is necessary to recover the phase data that are related to these magnitudes.

Techniques for phase recovery such as Fourier based [3], phase stepping [4] or regularization [5, 6, 7], provide a non-continuous phase wrapped in the interval $(-\pi, \pi]$. This phase needs to be unwrapped as a step to carry out the measure process of physical magnitudes. It is common to find phase inconsistencies or noise that can make the unwrapping process a difficult task. The application of path dependent algorithms [8] improves the unwrapping process but does not always provide proper results. A robust alternative for many cases is the least-squares solution which is described in matrix form by Hunt [9]. Other robust algorithm to find a solution in the presence of path-integral phase inconsistencies using the cosine transform is that proposed by Ghiglia and Romero [10]. The methods above mentioned present long processing time and computational complexity that make them inconvenient for some applications. When the phase is smooth, the time of processing can be shortened by solving the phase unwrapping problem using a linear combination of local basis functions [11]. In this paper we propose a regular grid of a linear combination of gaussian basis functions (RBFs) applied to unwrap the phase. The algorithm proposed here sited the centre of the RBFs in a subset of regular-spaced pixels of the phase map image. The weights are described in a typical matrix formulation allowing the matrix inversion using direct methods [12].

In this paper we give, in section 2, the fundamentals of the phase unwrapping problem. We describe, in section 3, our proposed phase unwrapping technique. In section 4 some practical considerations to implement our algorithm are discussed. Numerical experiments are presented in section 5 and finally, in section 6, we summarize some conclusions.

2. Relationship between wrapped and unwrapped phases

Defining φ_r and ϕ_r that represents the observed phase wrapped and the real unknown phase unwrapped respectively, where $r = (x, y)$ is the vector in a discrete grid, the relationship between these two phases is stablished by

$$\varphi_r = \mathcal{W}\{\phi_r\} = \phi_r + 2\pi k_r \tag{1}$$

where $\mathcal{W}$ represents the wrapping operator and k_r a field of integers such that $\mathcal{W}\{\phi_r\} \in (-\pi, \pi]$. The wrapped discrete phase gradient field, $\Delta\varphi_r$, is defined as

$$\nabla\varphi_r = (\varphi_r - \varphi_s, \varphi_r - \varphi_t) \tag{2}$$

where $s = r - (1, 0)$ and $t = r - (0, 1)$ are contiguous horizontal and vertical sites respectively. We can also define the unwrapped discrete phase gradient field as $\nabla\phi_r = (\varphi_r - \phi_s, \phi_r - \phi_t)$. If the sampling theorem is fulfilled for these two discrete phase fields, the problem of the recovery ϕ_r from φ_r can be properly solved. The sampling theorem establishes that the distance between two fringes must be more than two pixels (the phase difference between two fringes is 2π). For phase, the sampling theorem is reached if the phase difference between two pixels is less than π. This is

$$\|\nabla\phi_r\| < \pi. \tag{3}$$

If this condition is satisfied, we can establish:

$$\nabla\phi_r = \mathcal{W}\{\nabla\varphi_r\} = (\mathcal{W}\{\varphi_r - \varphi_s\}, \mathcal{W}\{\varphi_r - \varphi_t\}) \tag{4}$$

$\mathcal{W}\{\nabla\varphi_r\}$ can be obtained from the observed field. From this equation, we see that ϕ_r can be achieved by two-dimensional integration of the vector field $\mathcal{W}\{\nabla\varphi_r\}$. This can be carried out by using a least-squares approach [13, 14, 15].

3. Sampled radial basis functions for phase unwrapping

Any function can be modeled by a linear combination of basis functions. Let $U = \{0 \ldots M-1\} \subset \mathbb{Z}$ y $V = U \times U$ and considering the unwrapped phase $\phi : V \mapsto \mathbb{R}$ be smooth, it can be approximated by a linear combination of N local RBFs, this is

$$\phi_r \approx \sum_{i=1}^{N} w_i \psi(\|r - r_i\|) = \sum_{i=1}^{N} w_i \psi_r^i, \tag{5}$$

where $r \in V$, $w_i \in \mathbb{R}$ $(i \in \{1, \ldots, N\})$ are the weights of the N shifted basis functions $\psi(\|r - r_i\|) = \psi_r^i$ and $r_i \in \mathbb{R}^2$ are the central point of the RBFs. We use $U_h \subset U$ and $U_v \subset U$ are such as

$$|U_h|, \; |U_v| \leq M, \tag{6}$$

to express the horizontal points $V_h = U_h \times U$ and vertical points $V_v = U \times U_v$ of the grid used to recover the phase.

The horizontal and vertical finite differences $\Delta_s^h \phi_s$ and $\Delta_t^v \phi_t$ are given by

$$\begin{aligned} \Delta_s^h \phi_s &= \phi_{s+\iota} - \phi_s \\ \Delta_t^v \phi_t &= \phi_{t+\kappa} - \phi_t \end{aligned} \tag{7}$$

where $s \in U_h \times \{0 \ldots M-2\}$, $t \in \{0 \ldots M-2\} \times U_v$, $\iota = (1,0)$, $\kappa = (0,1)$. These finite differences can also be expressed in a similar way to Ec. (5):

$$\Delta_s^h \phi_s = \Delta_s^h \sum_{i=1}^{N} w_i \psi_s^i = \sum_{i=1}^{N} w_i \Delta_s^h \psi_s^i = \sum_{i=1}^{N} w_i \Psi_s^i \tag{8}$$

$$\Delta_t^v \phi_t = \Delta_t^v \sum_{i=1}^{N} w_i \psi_t^i = \sum_{i=1}^{N} w_i \Delta_t^v \psi_t^i = \sum_{i=1}^{N} w_i \Psi_t^i \tag{9}$$

where $\Psi_s^i = \Delta_s^h \psi_s^i$ and $\Psi_t^i = \Delta_t^v \psi_t^i$. Defining

$$\Theta_h^i = \begin{pmatrix} \Psi_{(k,0)}^i \\ \Psi_{(k,1)}^i \\ \Psi_{(k,2)}^i \\ \vdots \\ \Psi_{(k,M-2)}^i \end{pmatrix} \quad \Theta_v^i = \begin{pmatrix} \Psi_{(0,l)}^i \\ \Psi_{(1,l)}^i \\ \Psi_{(2,l)}^i \\ \vdots \\ \Psi_{(M-2,l)}^i \end{pmatrix} \tag{10}$$

where $k \in U_h$, $l \in U_h$. The phase unwrapping problem can be expressed in matrix form as

$$\Phi_d = \Theta \mathbf{w} \tag{11}$$

where Φ_d is the vector of unwrapped phase differences and

$$\Theta = \begin{pmatrix} \Theta_h^1 & \Theta_h^2 & \Theta_h^3 & \ldots & \Theta_h^N \\ \Theta_v^1 & \Theta_v^2 & \Theta_v^3 & \ldots & \Theta_v^N \end{pmatrix} = \begin{pmatrix} \Theta_h \\ \Theta_v \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_N \end{pmatrix} \tag{12}$$

For horizontal and vertical wrapped finite differences $\Delta_s^h \varphi_s$ and verticales $\Delta_t^v \varphi_t$ (related to phase φ) we express

$$\begin{aligned} \mathcal{W}\left\{\Delta_s^h \varphi_s\right\} &= \mathcal{W}\left\{\varphi_{s+\iota} - \varphi_s\right\} = \varphi_s^h \\ \mathcal{W}\left\{\Delta_t^v \varphi_t\right\} &= \mathcal{W}\left\{\varphi_{t+\kappa} - \varphi_t\right\} = \varphi_t^v \end{aligned} \tag{13}$$

and using matrix notation:

$$\Omega_h = \begin{pmatrix} \varphi_{(k,0)} \\ \varphi_{(k,1)} \\ \varphi_{(k,2)} \\ \vdots \\ \varphi_{(k,M-2)} \end{pmatrix}, \quad \Omega_v = \begin{pmatrix} \varphi_{(0,l)} \\ \varphi_{(1,l)} \\ \varphi_{(2,l)} \\ \vdots \\ \varphi_{(M-2,l)} \end{pmatrix} \text{ y } \ \Omega = \begin{pmatrix} \Omega_h \\ \Omega_v \end{pmatrix} \tag{14}$$

Given that $\mathcal{W}\left\{\Delta_s^h \varphi_s\right\} = \Delta_s^h \phi_s$ and $\mathcal{W}\left\{\Delta_t^v \varphi_t\right\} = \Delta_t^v \phi_t$, we can express the residual ρ as

$$\rho = \begin{pmatrix} \Omega_h \\ \Omega_v \end{pmatrix} - \begin{pmatrix} \Theta_h \\ \Theta_v \end{pmatrix} \mathbf{w} = \Omega - \Theta \mathbf{w} \tag{15}$$

Minimizing the norm of the residual ρ

$$\min_{\mathbf{w}} \|\rho\| = \min_{\mathbf{w}} \|\Omega - \hat{\Omega}\|^2 = \min_{\mathbf{w}} \|\Omega - \Theta \mathbf{w}\|^2 \tag{16}$$

we obtain the optimal $\mathbf{w}^*$.

4. Spatial distribution of RBFs for phase recovering

The spatial distribution of the RBFt́s was chosen regular as shown in Fig. **1**. For spatial distributions with $n \times n$ RBFs, the width σ for each RBF is given by

$$\sigma = \frac{M}{(n-1)\, d_{sg} + 2d_o} \tag{17}$$

where M is the width of the wrapped phase, d_{sg} is the distance between two nearest RBF's and d_o is the distance of the RBF to the border of the image. Figure **1** shows the centre of the RBF's. It is important to take into account that there must be at least three horizontal and vertical pixel lines within the influence of the RBF's to achieve the reconstruction.

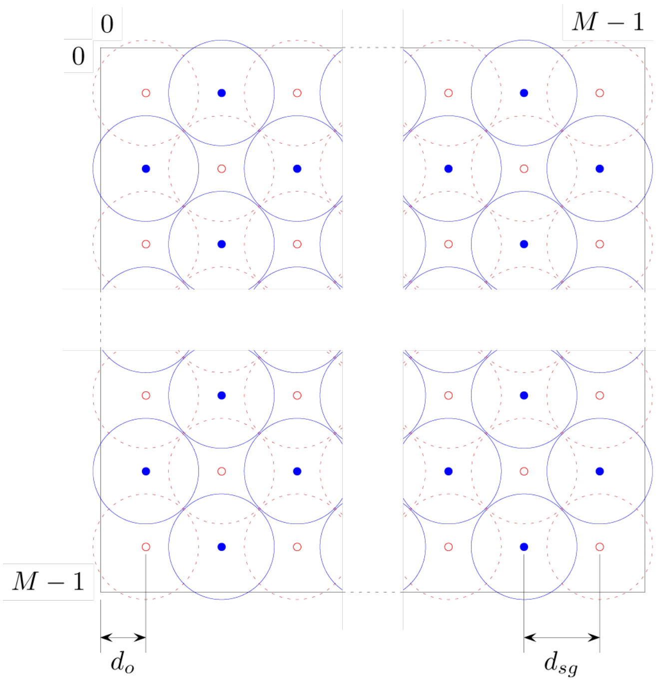

Figure 1: Regular distribution of RBF's within V. Their centres are represented by dots (filled and unfilled) and their influence area by circles (solid and dashed).

Tipically, the gaussian function is used as radial function. It is expressed as

$$G(\mathbf{x}) = \exp\left(-\frac{\|\mathbf{x}\|^2}{2\sigma^2}\right) \tag{18}$$

where $\mathbf{x} \in V$

5. Experimental results

In this section we present some experimental results using sampled radial basis function for phase recovery using the algorithm proposed in this paper and that reported by [11] for comparison purposes. The algorithms were tested with synthetic noiseless, synthetic noisy and real wrapped phase fields. Our algorithm used a spatial RBFs distribution shown in Figure **1**. In the first experiment we simulated a noiseless wrapped phase with $M = 512$, Figures **2**(a)-(b). In the Figures **2**(c)-(d) we show the phase unwrapped map with $n = 10$, $d_{sg} = 1$, $d_o = 0$. In Table **1** we show a summary with several parameters of interest. We proved our method with the same phase field but adding Gaussian noise with *zero mean* and *standard deviation* of 0.8. We show the errors in the reconstruction in Table **2**. In the second experiment we used a synthetic noisy phase map. Figure **3** shows the unwrapped phase maps using Villa *et al* [11] and our proposed method.

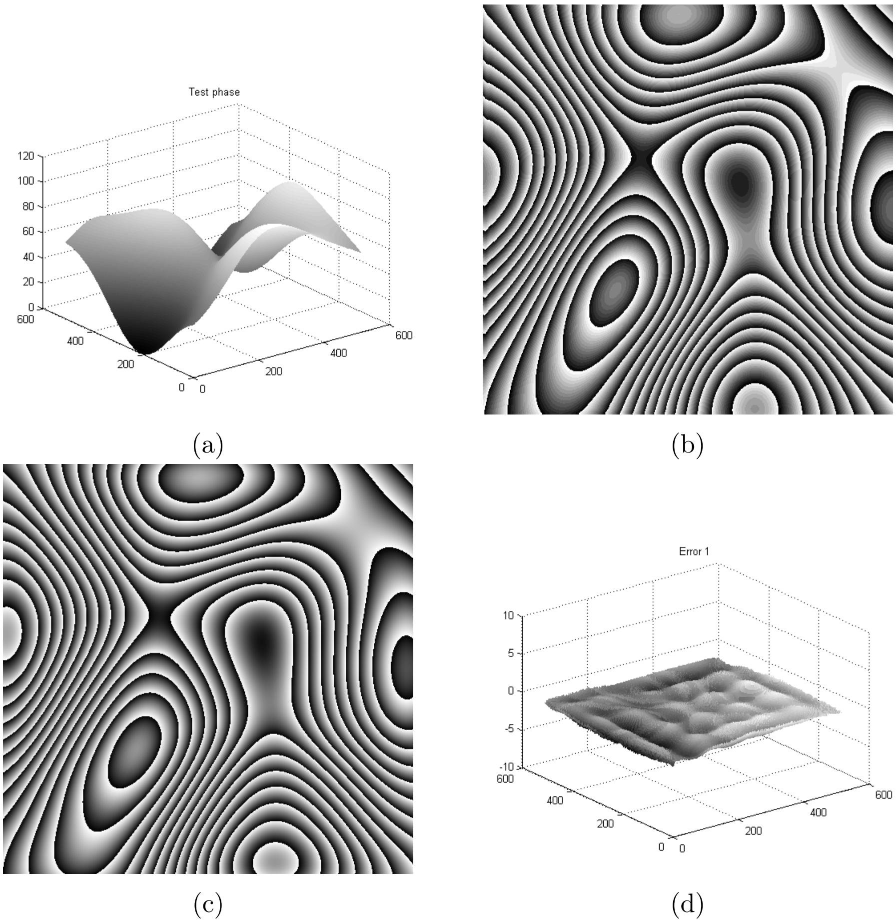

Figure 2: Phase maps. The space between horizontal and vertical lines are then. (a) Syntectic phase map. (b) Wrapped phase map (c) Unwrapped phase map using our proposed method. (d) Error for the unwrapped phase map.

	Memory (doubles)			Error			
Method	Ω	Θ	t (seg)	*rms*	*norm*	max	min
Villa	52326400	523264	8.355	0.423	126.192	1.184	-1.503
Proposed Algorithm	5314400	53144	0.945	0.421	123.426	1.205	-1.449

Table 1: Comparative table for Figure **2**.

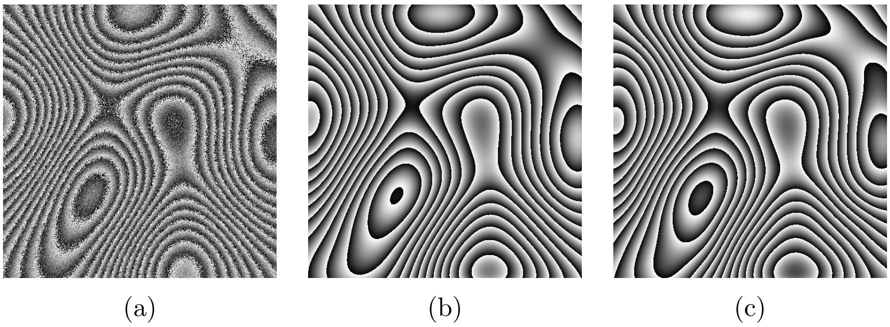

Figure 3: (a) Noisy Phase map. (b) Unwrapped phase using the method by Villa *et al* (c) Unwrapped phase using our proposed method

		Error			
Method	t (seg)	*rms*	*norm*	max	min
Villa	8.099	2.824	5038.868	5.553	-8.831
Proposed Algorithm.	0.921	2.828	5171.200	6.364	-9.361

Table 2: Comparative table for Figure **3**.

6. Conclusions

We have presented an algorithm based on sampled gaussian radial basis functions to recover the phase from a wrapped phase map. This algorithm has good performance when working with smooth phase maps with low level of noise. The sampling of the phase allows to decrease both the processing time and the memory resource required to unwrap the phase when compared to Villa *et al* [11].

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