# BOOTSTRAP METHODS APPLIED TO INDIRECT MEASUREMENT

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**Abstract**

A biased bootstrap technique is presented to obtain robust parameter and measurement estimates. Moreover, the estimation of a measurement probability density function (pdf) using classical bootstrap techniques is presented as our final goal. Most of the time, large scale repetition of an experiment is not economically feasible, the Monte Carlo method cannot be used for uncertainty characterization and bootstrap methods are proved to be a potentially useful alternative. The measurement characterization is driven by the pdf estimation in a non-linear non-Gaussian case and with limited observed data.

## I. Introduction

In many industrial applications, direct access to a measurement ($\boldsymbol{m}$) is not possible, so the measurement process must be considered as an inverse problem [7], since the measurement estimation is needed. The characterization of all statistical knowledge upon this quantity of interest is naturally derived by the probability density function (pdf) $\wp(\boldsymbol{m})$. Moreover, it is well known in practice that every observation (in a set of data) does not play the same role for determining estimates, tests or other statistics. Then, a robust nonlinear regression strategy called biased bootstrap [9, 10] is used for parameter and measurement estimation in presence of outliers. The aim of this article is then to appraise the pdf using bootstrap techniques and to compare the results with others obtained using a primitive Monte Carlo (PMC) technique. In such a problem, difficulties are typically encountered when the size of the random sample is small (and so asymptotic methods do not apply) or more generally when the distribution of the statistic cannot be analytically expressed. Most of the time, large scale repetition of an experiment is not economically feasible, the Monte Carlo method can not be used and bootstrap methods are proved to be a potentially useful alternative. Then, the idea to use bootstrap methods to access the pdf $\wp(\boldsymbol{m})$ is found to be attractive.

The bootstrap techniques were introduced by Efron [5] and have been mainly developed for the estimation of confidence intervals where few data are available [4, 6, 11]. Zoubir [15] gives a wide application of bootstrap techniques in signal processing and its potentially usefulness. The characterization of the measurement has been introduced for a nonlinear Gaussian framework [1, 2], the problem of pdf estimation in a more suitable or realistic framework (nonlinear non-Gaussian and limited observed data) is considered in this paper. The section II presents the general problem of measurement estimation, the procedure biased bootstrap for robust parameter and measurement estimation is described jointly with classical bootstrap (non-parametric) to assessing uncertainty characterization in section III, section IV describes a simple variance reduction technique for the classical bootstrap procedure. A simple nonlinear parametric model is used in section V to qualify the bootstrap performance and results, and finally some conclusions are given in section VI.

## II. Problem statement

A measurement can be defined as the best way to take advantage of the information given by the observed data $\boldsymbol{y}$. The first step of a measurement procedure consists in modeling the physical phenomenon in concern. Therefore, building a model becomes a goal on its own. In many applications, unknown quantities $\boldsymbol{m}$ have to be estimated from a vector of observed values $\boldsymbol{y}$. This may be encountered in several domains such as non destructive testing or so-called indirect measurement. It is due to the inability to use transducers to measure $\boldsymbol{m}$ directly for any reason of harsh environment, long distance or other. Measurement systems can be formalized by two equations [7] :

**i ).** The *Observation Equation* is described by the classical nonlinear regression model :

$$y_i = f(x_i, \boldsymbol{\theta}) + e_i, \quad i = 1, \ldots, n \tag{1}$$

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Where $x_i$ is an experimental design vector and under consideration that function $f(\cdot)$ is a known model of the unknown parameter vector $\boldsymbol{\theta}$. Some fitting technique to estimate $\boldsymbol{\theta}$ can be used, for example nonlinear least squares (NLS), maximum of likelihood (ML), M-estimator [12], etc.

**ii ).** The *Measurement Equation* (a nonlinear function of the parametric model),

$$\boldsymbol{m} = g(\boldsymbol{\theta}) \qquad (2)$$

The measurement is usually defined by a functional of the parametric model $\boldsymbol{m} = \psi(f(\cdot))$ (i.e. derivation, integration, interpolation, extrapolation, etc.). This relation is then transformed into a function of the parameters $\boldsymbol{\theta}$ such as in equation (2). It is supposed that the measurement depends on at least one of the parameters $\exists k \; s.t. \; \partial g / \partial \theta_k \neq 0$. The final goal is therefore an accurate statistical characterization of the measurement quantity $\boldsymbol{m}$.

## III. Bootstrap in nonlinear regression and measurement

The first goal is then to determine the *sampling distribution* $\hat{\wp}_{\hat{\boldsymbol{\theta}}}$. Efron [5], Freedman [8], Hinkley [11] and Wu [14] introduce and discuss many of the properties of the classical bootstrap method in regression analysis. When assumptions over the errors $e_i$ pdf are limited (for example the error random variables belong to an iid sequence, but the law is unknown), bootstrap techniques can help to approximate the errors pdf. These techniques are based on resampling a vector of a few observed residuals, then a Monte Carlo procedure gives an estimation of the parameter pdf, and finally the pdf of the measurement can be approximated. The first step for estimating the empirical density $\hat{\wp}_e$ is to work with residuals. The residuals are computed on the basis of observed data $y_i$ and the proposed model $f(\cdot)$. A fitting technique is used to obtain the estimation of $\boldsymbol{\theta}$. In fact robust nonlinear regression deals with outlier accommodation. To accommodate points with large residuals, a biased bootstrap technique that assures robustness has been used.

### III.1. Biased bootstrap parameter estimator

A biased bootstrap empirical method proposed in [10] is used for robust parameter estimation. Such a method identifies and downweights those data values that exert undue influence on a statistical estimator. The selected weights arise as resampling probabilities in a version of the weighted bootstrap and lead to a biased version of the uniform bootstrap. This approach does not need density estimation or the specification of a parametric family of distributions. The biased bootstrap requires two inputs : a distance measure between the uniform and the biased bootstrap distributions, and some constraints. The distance measure (e.g. power divergence) used, is proposed in [10] and is given by $D_\rho = \xi$, where $\xi$ is given by the following breakdown function :

$$\xi(\varepsilon) = \begin{cases} [\rho(1-\rho)]^{-1}\left\{1-(1-\varepsilon)^{1-\rho}\right\}, & \text{if } \rho \neq 1, \\ -\log(1-\varepsilon), & \text{if } \rho = 1, \end{cases} \qquad (3)$$

and is considered that it begins with a fixed value, given an initial breakdown point $\varepsilon$ in the range $0 < \varepsilon < 1/2$ (in practical examples it has been suggested $\varepsilon \in (0.01, 0.15)$), $\rho$ is the exponent of the power divergence and it varies in the range $0 < \rho \leq 1$, which includes Hellinger and Kullback-Leibler distances. The constraints are given by the measures of location and dispersion. The multivariate location estimation for a given bivariate sample $\{X_i, Y_i\}$, is obtained by equation (4) :

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} w_i \varphi(e_i) \qquad (4)$$

where $w_i$ are the weights and $\varphi(e_i)$ can be the quadratic norm function (BBQ) :

$$\varphi_1(e_i) = (y_i - f(x_i, \boldsymbol{\theta}))^2 \qquad (5)$$

or the robust Huber-like norm proposed in [2] and given by the following equation (BBH) :

$$\varphi_2(e_i) = \frac{\Delta^2}{2}\left(\sqrt{1 + \frac{4\varphi_1(e_i)}{\Delta^2}} - 1\right), \qquad (6)$$

where $\Delta > 0$ and is a constant value. The measure of dispersion is given by :

$$\gamma(w_i) = \inf_{\boldsymbol{\theta}} \sum_{i=1}^{n} w_i \varphi(e_i) . \qquad (7)$$

The usual empirical distribution is given by the uniform bootstrap and considering that all weights have the same mass in all data points $\boldsymbol{w}_{unif} = (1/n, \ldots, 1/n)$. The initial values estimated for location and dispersion are given as follows $\hat{\theta} = \hat{\theta}(\boldsymbol{w}_{unif})$ and $\hat{\gamma} = \gamma(\boldsymbol{w}_{unif})$, then the level of dispersion is calibrated (e.g. minimized) by the biased weights (e.g. downweights) $w_i$. The biased weights function proposed in [10] is given in the following general form :

$$w_i = \frac{1}{n} h_{[\xi]}\left(X_i, Y_i \setminus \hat{\boldsymbol{\theta}}, \hat{\tau}, \hat{\lambda}\right) \qquad (8)$$

where

$$h_{[\xi]}(\cdot)=\begin{cases}\left(1+(\rho-1)\left[\rho\xi-\hat{\lambda}\{\varphi(\hat{e}_i)-\hat{\tau}\}\right]\right)^{1/(\rho-1)}, & \\ & \text{if } \rho\neq 1,\\ \exp\left(\xi-\hat{\lambda}\{\varphi(\hat{e}_i)-\hat{\tau}\}\right), & \\ & \text{if } \rho=1.\end{cases}$$

The values given by $(\hat{\boldsymbol{\theta}},\hat{\tau},\hat{\lambda})$ satisfy the following equations :

1 ) .- criterion minimization $\frac{\partial}{\partial\boldsymbol{\theta}}\left[\sum_{i=1}^{n}w_i\varphi(\hat{e}_i)\right]=0$,

2 ) .- calculation of $\hat{\tau}=\sum_{i=1}^{n}w_i\varphi(\hat{e}_i)$, where $0<\hat{\tau}\leq\hat{\gamma}$,

3 ) .- $\hat{\lambda}$ chosen by dichotomy given that :

$$\sum_{i=1}^{n}w_i=1.$$

If the distribution of $e_i$ is symmetric and unimodal, the *true* values of $\boldsymbol{\theta}$ are not changed by trimming, and in the asymmetric case they are altered, although they remain well defined as solution of $\boldsymbol{\theta}\in\boldsymbol{\Theta}\subset\Re^p$.

## III.2. Parameter pdf estimation

The conventional uniform bootstrap methods can be used to approximate the distribution of the biased estimator. The *simple* residuals $\hat{e}_i$ are therefore obtained by using the following expression :

$$\hat{e}_i=y_i-f\left(x_i,\hat{\boldsymbol{\theta}}\right) \qquad (9)$$

The sample probability distribution of residuals $\hat{\wp}_e$ is then approximated by punctual statistical mass of $1/n$ at each realization of $\hat{e}_i$.

The next step is to draw the bootstrap samples $\hat{e}_i^*$ and $y_i^*$, given $\hat{e}_i\sim\hat{\wp}_e$ and $\hat{\boldsymbol{\theta}}$:

$$y_i^*=f\left(x_i,\hat{\boldsymbol{\theta}}\right)+\hat{e}_i^* \qquad \hat{e}_i^*\sim\hat{\wp}_e \qquad (10)$$

Indeed, $\hat{\boldsymbol{\theta}}$ will be assumed as the *true* parameters. Each realization of $\hat{e}_i^*$ yields an estimation of $\hat{\boldsymbol{\theta}}^*$ by the same minimization process that gave $\hat{\boldsymbol{\theta}}$ for example equation (4).

Repeating $B$ independent bootstrap replications for $\hat{\boldsymbol{\theta}}^*$ will give a random sample $\hat{\boldsymbol{\theta}}^{*1},\ldots,\hat{\boldsymbol{\theta}}^{*B}$, which can be used to estimate the bootstrap distribution of $\hat{\boldsymbol{\theta}}^*$ and then the pdf of $\hat{\boldsymbol{\theta}}$ could be approximated by $\wp\left(\hat{\boldsymbol{\theta}}^*\right)$.

The bootstrap performance and approximation have been considered by working with residuals for both Gaussian and non-Gaussian “assumptions”. The level of approximation can be improved when some different *priors* are incorporated to the residuals vector under symmetric assumptions :

**a)** *centered* residuals, if one of the components of $\boldsymbol{\theta}$ is a translation parameter for the function $f(\cdot)$ then $\hat{\wp}_e$ has zero mean. If not, $\hat{\wp}_e$ might be still modified by translation to achieve zero mean [5, 8],

**b)** and *modified* residuals, bootstrapping modified residuals gives a *smoothed* (or *weighted*) version of residuals that can drive to a consistent statistics estimator (e.g. weighted bootstrap)[13].

## III.3. Measurement pdf estimation

An extended measurement vector is given by $\boldsymbol{m}=\left[\theta_1,\ldots,m,\ldots,\theta_p\right]^{\mathsf{T}}$. It gives the nonlinear mapping $\boldsymbol{m}=G(\boldsymbol{\theta})$. Once the parameters estimate $\hat{\boldsymbol{\theta}}^*$ is computed, the measurement induced by the mapping $G$ is given as :

$$\hat{\boldsymbol{m}}^*=G\left(\hat{\boldsymbol{\theta}}^*\right) \qquad (11)$$

The measurement pdf $\wp_m(\boldsymbol{m})$ is then approximated by the bootstrap measurement pdf $\wp_m\left(\hat{\boldsymbol{m}}^*\right)$, which is induced by $\hat{\boldsymbol{m}}^{*1},\ldots,\hat{\boldsymbol{m}}^{*B}$ using the different bootstrap replications $\hat{\boldsymbol{\theta}}^{*1},\ldots,\hat{\boldsymbol{\theta}}^{*B}$ in the nonlinear mapping $G$.

## IV. Nested bootstrap

In the literature, it is usual to see that the number of bootstrap simulation replications $B$ is approximately the same value as the sample size $n$. In parametric models with sufficient regularity, asymptotically statistics, which have a limiting distribution that does not depend on unknown parameters, the bootstrap will yield an approximation that is closer to the true distribution, in terms of orders of approximation to the probability. By nesting the original bootstrap within another bootstrap, the approximation error can be similarly reduced. In other words, nested bootstrap (e.g. simulated bootstrap) can be used to further reduce the time of computations.

To dominate the stochastic error introduced by the bootstrap unless $B\geq n^2$. In a simulated nested bootstrap, the number of replications in the inner bootstrap requires $B_1 \geq n^2$, but the number of replications in the outer bootstrap must satisfy $B_2 \geq n^3$. Thus the total number of replications must be $n^5$ to insure that the simulation stochastic error is not so large. This computational requirement can be prohibitive even in simple models with moderate samples. A control variate approach can therefore be implemented for the bootstrap pdf approximation. This approach uses a leading term in the asymptotic expansion for the statistic of interest and its expectation to form a function that has the same expectation but smaller variance than the original statistic. The number of simulations required in the inner loop of nested bootstrap is reduced to $B_1 \geq n$, and for the outer loop $B_2 \geq n^2$. Thus the total number of replications required is reduced to $n^3$. Although the computational requirements involved in the later modification can be substantial, and it is a good improvement over the requirements for a standard nested bootstrap. This approach seems to give good results, when the underlying distribution of the variables of interest is parametrically specified. Its extension to semiparametric models, such as nonlinear regression and measurement has been considered, and where the parameters $\boldsymbol{\beta} = [\boldsymbol{\theta}, \boldsymbol{m}]^{\mathrm{T}}$. A first approach was introduced in [3] for nonlinear regression.

The distribution function of any statistic $T(\boldsymbol{\beta})$ of $\boldsymbol{\beta}$ is given by $\Gamma(\boldsymbol{\beta}, T(\boldsymbol{\beta})) = E_{\beta}\{1(T(\boldsymbol{\beta}) \leq t)\}$ [3], and the function of interest is $H(T(\boldsymbol{\beta}), t) = 1(T(\boldsymbol{\beta}))$, it is necessary to assume that $T(\boldsymbol{\beta})$ has a finite sample knowledge of a leading term, presented here in its *Edgeworth* expansion form :

$$T(\boldsymbol{\beta}) = \omega(e_1, \ldots, e_n; \boldsymbol{\beta}) + O_p\left(n^{-q/2}\right) \qquad (12)$$

where $\omega(\boldsymbol{\beta}) = \omega(e_1, \ldots, e_n; \boldsymbol{\beta})$ is an approximated statistic of $T(\boldsymbol{\beta})$, given the errors random sample $e_i \sim \hat{\wp}_e(\hat{\boldsymbol{\theta}})$, from its known parameterized density, and $F_{\omega}(\boldsymbol{\beta}, t) = E_{\beta}\{1(\omega(\boldsymbol{\beta}) \leq t)\}$ are known up to the parameters. The statistic $F_{\omega}(\boldsymbol{\beta}, t)$ is pivotal and does not depend on $\boldsymbol{\beta}$ when $q = 1$, then $F_{\omega}(\boldsymbol{\beta}, t) = F_{\omega}(t)$. The knowledge of $T(\boldsymbol{\beta})$ and $H(T(\boldsymbol{\beta}), t)$ is used to form a control variate that eliminates the leading stochastic term in $H(T(\boldsymbol{\beta}), t)$, by simulating the reminder.

Given $\boldsymbol{\beta}$ and knowledge of $\omega(\boldsymbol{\beta})$, one generates $e_i^{(j)} \sim \hat{\wp}_e(\hat{\boldsymbol{\theta}})$, $j = 1, \ldots, B_1$ forming $T^{(j)}(\boldsymbol{\beta})$ and $\omega^{(j)}(\boldsymbol{\beta}) = \omega(e_i^{(j)}; \boldsymbol{\beta})$, and defining :

$$\widetilde{H}\left(T^{(j)}(\boldsymbol{\beta}), \omega^{(j)}(\boldsymbol{\beta}), t\right) = 1\left(T^{(j)}(\boldsymbol{\beta})\right) - 1\left(\omega^{(j)}(\boldsymbol{\beta})\right) + F_{\omega}(\boldsymbol{\beta}, t) \qquad (13)$$

and its average over the Monte Carlo is :

$$\widetilde{\Gamma}_{B_1}(\hat{\boldsymbol{\beta}}, t) = \frac{1}{B_1} \sum_{j=1}^{B_1} \widetilde{H}\left(T^{(j)}(\hat{\boldsymbol{\beta}}), \omega^{(j)}(\hat{\boldsymbol{\beta}}), t\right)$$

$$\widetilde{\Gamma}_{B_1}(\hat{\boldsymbol{\beta}}, t) = \Gamma(\boldsymbol{\beta}_0, t) + O_p\left(n^{-(k+1)/2}\right) + O_p\left(n^{-q/2} B_1^{-1/2}\right) \qquad (14)$$

It is required that $B_1 \geq n^{k+1-q}$ to ensure that Monte Carlo error does not dominate. In the outer bootstrap $B_2$ replications are necessary, and the function of interest is given by $\widetilde{T}_1(\boldsymbol{\beta}) = \widetilde{\Gamma}_{B_1}(\hat{\boldsymbol{\beta}}, T(\boldsymbol{\beta}))$, where $\widetilde{T}_1(\boldsymbol{\beta}) = \Gamma(\boldsymbol{\beta}, T(\boldsymbol{\beta})) + O_p\left(n^{-(k+1)/2}\right)$, and distribution function given by $\widetilde{\Gamma}_1(\boldsymbol{\beta}, t_1) = E_{\beta}\left\{1\left(\widetilde{T}_1(\boldsymbol{\beta}) \leq t_1\right)\right\}$.

Applying the previous ideas, one defines :

$$\widetilde{H}_1\left(\widetilde{T}_1^{(l)}(\boldsymbol{\beta}), \omega^{(l)}(\boldsymbol{\beta}), t_1\right) = 1\left(\widetilde{T}_1^{(l)}(\boldsymbol{\beta})\right) - 1\left(F_{\omega}\left(\omega^{(l)}(\boldsymbol{\beta})\right)\right) + t_1 \qquad (15)$$

and its average over the Monte Carlo is :

$$\widetilde{\Gamma}_{1B_2}(\hat{\boldsymbol{\beta}}, t) = \frac{1}{B_2} \sum_{l=1}^{B_2} \widetilde{H}_1\left(\widetilde{T}_1^{(l)}(\hat{\boldsymbol{\beta}}), \omega^{(l)}(\hat{\boldsymbol{\beta}}), t_1\right)$$

$$\widetilde{\Gamma}_{1B_2}(\hat{\boldsymbol{\beta}}, t) = \widetilde{\Gamma}_1(\boldsymbol{\beta}_0, t_1) + O_p\left(n^{-(k+2)/2}\right) + O_p\left(n^{-q/2} B_2^{-1/2}\right) \qquad (16)$$

In general, it is required that $B_2 \geq n^{k+2-q}$, then for a common case $k = q = 1$, $B_1 \geq n$ and $B_2 \geq n^2$ and the total replications are $n^3$. Some restrictions are given for the errors pdf assumptions, one needs independence of $x_i$, zero mean (symmetry) and constant variance. When drawing $e_i^{(j)} \sim \hat{\wp}_e$ from its empirical distribution, there are complications to find $F_{\omega}(\boldsymbol{\beta}, t)$, but it should be possible to work out of its expectation function by giving a number of additional Monte Carlo replications.

## V. Example

Some experimental results based on a nonlinear model have been obtained. The goal is to compare the asymptotic pdf with the bootstrap approximation when hypothesis of either Gaussianity or non-Gaussianity over the errors has been made. Some improvement suggestions have been implemented. The nonlinear proposed model is the following one :

$$f(x_i, \boldsymbol{\theta}) = \theta_1 \tanh(\theta_2 x_i)$$

where $x_i \in [0, 1, \ldots, 15]^{\mathrm{T}}$ and $\boldsymbol{\theta}_0 = [2.18, 0.49]^{\mathrm{T}}$. The observation vector $y_i$ is used to estimate the slope at the origin $\partial f / \partial x(0,\boldsymbol{\theta})$, so $m$ can be analytically expressed by :

$$m = g(\boldsymbol{\theta}) = \theta_1\theta_2 .$$

Then the typical measurement is $m_0 = 1.0682$. To evaluate the estimation performance, the probability density of $m$ has been derived. Bootstrap approximation and PMC asymptotic results determine pdf estimations of the parameters and the measurement. The results when the hypothesis upon $e_i \sim \mathbf{N}\left(0,\sigma^2\right)$, $e_i \sim \mathbf{U}(0,\alpha)$ and the mixture $e_i \sim \mathbf{N}\left(0,\sigma^2\right) + \mathbf{U}(-\alpha/2,\alpha/2)$ (non-Gaussian symmetric case) are studied for $1e5$ PMC (asymptotic pdf) vs. different bootstrap replications $B$ ($\sigma^2 = 0.0625$ and $\alpha = \sigma$). The finality is to quantify the level of approximation produced by the different bootstrap schemes. The mean, the bias and the variance observed by PMC are presented in contrast with best approximations obtained by bootstrap (modified residuals) and nested bootstrap (NB), when $e_i$ is a mixture noise, see table 1. On the other hand, figure 1 shows the asymptotic convergence of the bootstrap pdf. The asymptotic PMC reference pdf, is used to evaluate the distance between $\wp_r(\boldsymbol{m})$ and $\wp_m(\hat{\boldsymbol{m}}^*)$. Some different distance metric functions are recommended in [2], the Hellinger distance is preferred here :

$$\Delta_H\left(\wp_r(\cdot),\wp_m(\cdot)\right) = \left(\int\left(\sqrt{\wp_r(\boldsymbol{m})} - \sqrt{\wp_m(\hat{\boldsymbol{m}}^*)}\right)^2 dm\right)^{1/2}$$

It is used to figure the level of the bootstrap approximation versus the PMC asymptotic pdf.

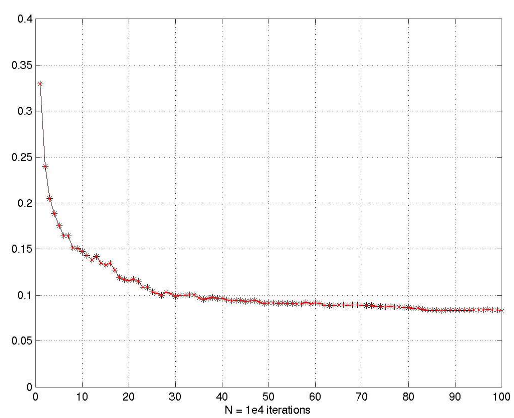


**Fig. 1**. Distance between pdf approximated by bootstrap (modified residuals) and asymptotic PMC reference pdf for a measurement in symmetric noise assumptions.

Robust estimation using the biased bootstrap idea is presented. In this case, the assumptions of the noise are $e_i \sim \mathbf{U}(0,\alpha)$ plus outliers. The figure 2 depicts the obtained results, when the quadratic norm (BBQ) and the Huber-like norm (BBH) are used. The NLS estimator gives the poorest approximation in the fitting sense. The values for $\Delta, \rho, \lambda$ and $\varepsilon$ were 0.25, 1, 0.25 and 0.0165 respectively.

**Table 1**. Bootstrap and Nested Bootstrap vs. PMC statistics for $\boldsymbol{\theta}$ and $m$, using the NLS estimator.

| | Mean $(\theta_1)$ | Bias $(\theta_1)$ | Var. $(\theta_1)$ |
|---|---|---|---|
| PMC 1e 5 | 2.1835 | 0.0035 | 0.0059 |
| BOOT 3000 | 2.1832 | 0.0032 | 0.0058 |
| BOOT 5000 | 2.1831 | 0.0031 | 0.0056 |

| | Mean $(\theta_2)$ | Bias $(\theta_2)$ | Var. $(\theta_2)$ |
|---|---|---|---|
| PMC 1e 5 | 0.5064 | 0.0164 | 0.0123 |
| BOOT 3000 | 0.5056 | 0.0156 | 0.0111 |
| BOOT 5000 | 0.5062 | 0.0162 | 0.0112 |

| | Mean $(m)$ | Bias $(m)$ | Var. $(m)$ |
|---|---|---|---|
| PMC 1e 5 | 1.1020 | 0.0338 | 0.0511 |
| BOOT 3000 | 1.1007 | 0.0325 | 0.0464 |
| BOOT 5000 | 1.1019 | 0.0337 | 0.0471 |
| n = 12 | | | |
| NB 1728 | 1.0948 | 0.0266 | 0.0432 |
| n = 16 | | | |
| NB 4096 | 1.1055 | 0.0373 | 0.0482 |

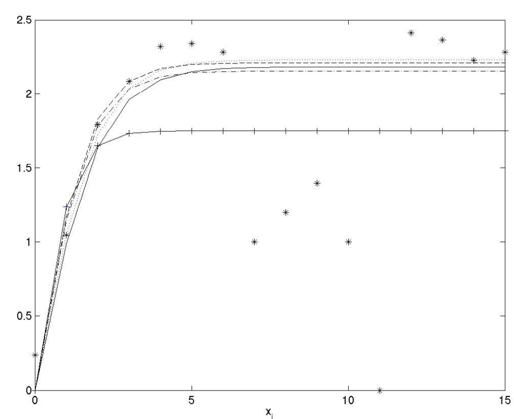


**Fig. 2**. Comparison between the different parameter estimators using their mean estimation value. Data generation Model (—), Data corrupted by uniform noise and outliers (∗), PMC – NLS (—+), PMC – BBH (--·-), Bootstrap – BBQ (---) and Bootstrap – BBH (····).

The figure 3 shows the different approximated pdfs, one can see that the best estimator in the measurement sense is the BBH, the typical measurement is the reference. Finally, table 2 presents some estimated statistics for the measurement, given by the different estimators previously compared.

**Table 2**. Measurement statistics, using the different estimators.

| $m_r = 1.0682$ | Mean $(m)$ | Bias $(m)$ | Var. $(m)$ |
|---|---|---|---|
| PMC – NLS | 1.5367 | 0.4685 | 0.0115 |
| PMC – BBH | 1.2813 | 0.2131 | 0.0049 |
| Boot – BBQ | 1.2678 | 0.1996 | 0.5257 |
| Boot – BBH | 1.0467 | 0.0215 | 0.6617 |

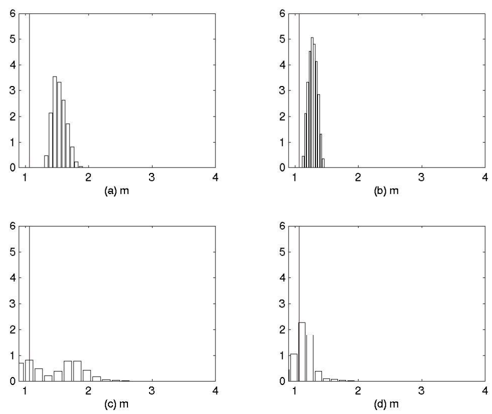


**Fig. 3**. Comparison between the measurement of reference (—) and the different estimated histograms : PMC – NLS (a), PMC – BBH (b), Bootstrap – BBQ (c) and Bootstrap – BBH (d).

## VI. Conclusions

The bootstrap results are very close to the PMC asymptotic results. Moreover, they could be improved using prior information on the residuals in a correct way. It must be important to control the bootstrap application in a careful way. Under Gaussian and non-Gaussian assumptions, the bootstrap gives a good approximation of the measurement pdf; it still works in the case of robust estimation. With a minimal number of iterations, the bootstrap approximation is very close to the PMC asymptotic estimation. The nested bootstrap, as a variance reduction technique, significantly reduces the number of iterations, but the variance is poorly reduced, and the only restrictions are the noise pdf symmetry assumption.

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