Electromagnetic potentials without gauge transformations

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Corrigendum

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Equations (H1) and (H2) should have been written

$$U(\mathbf{r}) = \frac{1}{4\pi}\int\int\int_{\text{All space}} \frac{D(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}' \qquad \text{(H1)}$$

and

$$\mathbf{W}(\mathbf{r}) = \frac{1}{4\pi}\int\int\int_{\text{All space}} \frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}'. \qquad \text{(H2)}$$

Equations (7) should have been written

$$\mathbf{j}_{\mathrm{i}} = -\frac{1}{4\pi}\nabla\int\int\int_{\text{All space}} \frac{\nabla'\cdot\mathbf{j}}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}'$$
$$\mathbf{j}_{\mathrm{s}} = \frac{1}{4\pi}\nabla\times\int\int\int_{\text{All space}} \frac{\nabla'\times\cdot\mathbf{j}}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}'. \qquad (7)$$

Equation (16), (18) and (19) should have been written

$$\varphi_j(x,y,z,t)_{\mathrm{i}} = \frac{1}{4\pi}\int\int\int_{\text{All space}} \frac{\nabla'\cdot\mathbf{j}_{\mathrm{i}}}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}' = -\frac{1}{4\pi}\int\int\int_{\text{All space}} \frac{\frac{\partial}{\partial t}\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}', \qquad (16)$$

$$\frac{\partial\Phi}{\partial t} = -4\pi\varphi_j = \int\int\int_{\text{All space}} \frac{\frac{\partial}{\partial t}\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}', \qquad (18)$$

$$\Phi = \int\int\int_{\text{All space}} \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}'. \qquad (19)$$

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Electromagnetic potentials without gauge transformations

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Abstract
In this paper, we show that the use of the Helmholtz theorem enables the derivation of uniquely determined electromagnetic potentials without the necessity of using gauge transformation. We show that the electromagnetic field comprises two components, one of which is characterized by instantaneous action at a distance, whereas the other propagates in retarded form with the velocity of light. In our attempt to show the superiority of the new proposed method to the standard one, we argue that the action-at-a-distance components cannot be considered as a drawback of our method, because the recommended procedure for eliminating the action at a distance in the Coulomb gauge leads to theoretical subtleties that allow us to say that the needed gauge transformation is not guaranteed. One of the theoretical consequences of this new definition is that, in addition to the electric **E** and magnetic **B** fields, the electromagnetic potentials are real *physical quantities*. We show that this property of the electromagnetic potentials in quantum mechanics is also a property of the electromagnetic potentials in classical electrodynamics.

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1. Introduction

In order to determine the electromagnetic field, we must find six values, which are the components of the electric **E** and magnetic **B** fields. In a number of cases, however, one can reduce this problem to finding four and sometimes a smaller number of values. With this aim, one introduces the field potentials vector **A** and scalar φ in the following way. Considering the Maxwell equations in a vacuum, for instance, in the Gauss system

$$\left.\begin{aligned} \nabla\cdot\mathbf{E} &= 4\pi\varrho \\ \nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{B} &= \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j} \end{aligned}\right\}, \qquad (1)$$

one can see that

$$\left.\begin{aligned} \mathbf{E} &= -\nabla\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla\times\mathbf{A} \end{aligned}\right\}. \qquad (2)$$

Indeed taking into account equation (2), one can reduce the system of eight equations (1) of six variables **E** and **B** to the system of four equations of four variables **A** and φ :

$$\left.\begin{aligned} \nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\frac{1}{c}\frac{\partial\varphi}{\partial t}+\nabla\cdot\mathbf{A}\right) &= -\frac{4\pi}{c}\mathbf{j} \\ \nabla^2\varphi+\frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\mathbf{A} &= -4\pi\rho \end{aligned}\right\}, \qquad (3)$$

The vector potential **A** and scalar potential φ (see equation (2)) thus introduced are not uniquely defined because they can be the subject of a gauge transformation. It would be very difficult to solve system (3), but it is known that the fields

E and **B** are invariant under the gauge transformation

$$\left.\begin{aligned}\mathbf{A}' &= \mathbf{A} + \nabla\psi \\ \varphi' &= \varphi - \frac{1}{c}\frac{\partial\psi}{\partial t}\end{aligned}\right\}, \tag{4}$$

where ψ is some arbitrary scalar function $\psi(x, y, z, t)$. One can choose ψ so as to impose on potentials **A** and φ an additional condition, for example

$$\nabla\cdot\mathbf{A} = -\frac{1}{c}\frac{\partial\varphi}{\partial t} \quad \text{or} \quad \nabla\cdot\mathbf{A} = 0, \tag{5}$$

which are usually known as the Lorentz gauge or the Coulomb gauge, respectively, and which permit us to separate equations (3) for these potentials.

It should be noted, however, that there are some problems in calculating the electromagnetic field of a moving particle in both the Lorentz and the Coulomb gauge (see e.g. [1, 2] and references therein). In the present work (the next section), we put forward a new approach, which saves us the trouble of *forced* use of either the Lorentz and Coulomb conditions or any other gauge conditions. This approach is founded on the use of the Helmholtz theorem to separate the electric field **E** on irrotational and solenoidal parts, in such a way that it can be expressed in the following functional form:

$$\mathbf{E} = -\nabla U + \nabla\times\mathbf{W}, \tag{HE}$$

where U and **W** are new potentials coming from the Helmholtz theorem even though conventional wisdom could say that the Helmholtz theorem does not introduce any kind of potentials. What is the difference between these potentials and the conventional ones? The answer is that the main difference lies in the fact that we can deduce, as we shall see in the next section, a pair of differential equations for these potentials, without the use of any gauge condition.

2. Applying the Helmholtz theorem to the vector fields in the Maxwell equations

In the generally accepted form, the Helmholtz vector decomposition theorem reads as follows (see e.g. section 1.16, p 92 in [3]): *If the divergence* $D(\mathbf{r})$ *and curl* $\mathbf{C}(\mathbf{r})$ *of a vector function* $\mathbf{F}(\mathbf{r})$ *are specified, and if they both go to zero faster than* $1/r^2$ *as* $r\to\infty$, *and if* $\mathbf{F}(\mathbf{r})$ *itself tends to zero as* $r\to\infty$, *then* $\mathbf{F}(\mathbf{r})$ *is uniquely given by*

$$\mathbf{F} = -\nabla U + \nabla\times\mathbf{W}, \tag{H}$$

where

$$U(\mathbf{r}) = \frac{1}{4\pi}\iiint_{\text{All space}}\frac{D(\mathbf{r}')}{|\mathbf{r}'-\mathbf{r}|}\mathrm{d}^3\mathbf{r}', \tag{H1}$$

and

$$W(\mathbf{r}) = \frac{1}{4\pi}\iiint_{\text{All space}}\frac{C(\mathbf{r}')}{|\mathbf{r}'-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}'. \tag{H2}$$

Let us apply the Helmholtz theorem (H) to the vector fields in the Maxwell equations (1), supposing that all conditions of this theorem are fulfilled for these fields (the validity of applying the Helmholtz theorem in the case of time-dependent vector fields is reasoned, for example, in [4–6]) and for almost all the vector functions that appear on the development because we shall use Helmholtz decomposition several times, so that

$$\left.\begin{aligned}\mathbf{E} &= \mathbf{E}_\mathrm{i} + \mathbf{E}_\mathrm{s} \\ \mathbf{B} &= \mathbf{B}_\mathrm{i} + \mathbf{B}_\mathrm{s} \\ \mathbf{j} &= \mathbf{j}_\mathrm{i} + \mathbf{j}_\mathrm{s}\end{aligned}\right\}, \tag{6}$$

where the indices 'i' and 's' signify irrotational (curl-less) and solenoidal (divergence-less) components of the vectors, respectively, and for example

$$\mathbf{j}_\mathrm{i} = -\frac{1}{4\pi}\nabla\iiint_{\text{All space}}\frac{\nabla\cdot\mathbf{j}}{|\mathbf{r}'-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}'$$

and

$$\mathbf{j}_\mathrm{s}\frac{1}{4\pi}\nabla\times\iiint_{\text{All space}}\frac{\nabla\times\mathbf{j}}{|\mathbf{r}'-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}'. \tag{7}$$

Substituting (6) into (1), we obtain the following equations for the irrotational part:

$$\left.\begin{aligned}\nabla\cdot\mathbf{E}_\mathrm{i} &= 4\pi\varrho \\ \frac{\partial\mathbf{E}_\mathrm{i}}{\partial t} &= -4\pi\mathbf{j}_\mathrm{i}\end{aligned}\right\}, \tag{8}$$

$$\left.\begin{aligned}\nabla\cdot\mathbf{B}_\mathrm{i} &= 0 \\ \frac{\partial\mathbf{B}_\mathrm{i}}{\partial t} &= 0\end{aligned}\right\}, \tag{9}$$

and the following for the solenoidal part:

$$\left.\begin{aligned}\nabla\times\mathbf{E}_\mathrm{s} &= -\frac{1}{c}\frac{\partial\mathbf{B}_\mathrm{s}}{\partial t} \\ \nabla\times\mathbf{B}_\mathrm{s} &= \frac{1}{c}\frac{\partial\mathbf{E}_\mathrm{s}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_\mathrm{s}\end{aligned}\right\}. \tag{10}$$

From the definition of $\mathbf{E}_\mathrm{i}$ one can write its relation to some scalar potential Φ as

$$\mathbf{E}_\mathrm{i} = -\nabla\Phi. \tag{11}$$

This, due to the first equation from (8), gives

$$\nabla^2\Phi = -4\pi\varrho. \tag{12}$$

The second equation from (8) is merely the law of conservation of charge. Indeed, if we take the divergence of two parts of this equation, we obtain

$$\frac{\partial}{\partial t}\nabla\cdot\mathbf{E}_\mathrm{i} = -4\pi\,\nabla\cdot\mathbf{j}_\mathrm{i}, \tag{13}$$

but $\nabla\cdot\mathbf{E}_\mathrm{i} = 4\pi\varrho$ and $\nabla\cdot\mathbf{j}_\mathrm{i} = \nabla\cdot(\mathbf{j}_\mathrm{i}+\mathbf{j}_\mathrm{s}) = \nabla\cdot\mathbf{j}$, so that equation (13) comes down to the well-known continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0. \tag{14}$$

At face value, it seems that equation (12) does not define the potential Φ completely, because there is another (temporal) differential equation for Φ. Indeed, from

$$\frac{\partial\mathbf{E}_\mathrm{i}}{\partial t} = -4\pi\mathbf{j}_\mathrm{i}, \quad \mathbf{E}_\mathrm{i} = -\nabla\Phi \quad \text{and} \quad \mathbf{j}_\mathrm{i} = -\nabla\varphi_j, \tag{15}$$

where φ_j, obviously, is the scalar function

$$\varphi_j(x,y,z,t)_{\rm i} = \frac{1}{4\pi}\iiint_{\text{All space}} \frac{\nabla\cdot\mathbf{j}_{\rm i}}{|\mathbf{r}'-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}' = -\frac{1}{4\pi}\iiint_{\text{All space}} \frac{\frac{\partial}{\partial t}\rho(\mathbf{r}',t)}{|\mathbf{r}'-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}' \quad (16)$$

(here we take into account equations (14) and (15) and $\nabla\cdot\mathbf{j}_{\rm i} = \nabla\cdot\mathbf{j}$), we obtain

$$-\nabla\frac{\partial\Phi}{\partial t} = -4\pi(-\nabla\varphi_j) \Rightarrow \frac{\partial\Phi}{\partial t} = -4\pi\varphi_j. \quad (17)$$

However, this equation, together with equation (16), comes down to

$$\frac{\partial\Phi}{\partial t} = -4\pi\varphi_j = \iiint_{\text{All space}} \frac{\frac{\partial}{\partial t}\rho(\mathbf{r}',t)}{|\mathbf{r}'-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}', \quad (18)$$

so that

$$\Phi = \iiint_{\text{All space}} \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}'-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}'. \quad (19)$$

But that coincides with the solution of the Poisson equation (12). Thus, we have ascertained that equation (12) with the corresponding boundary conditions defines the potential Φ completely, or more precisely, that, up to some irrelevant functions which, due to the boundary conditions, are reduced to zero.

System (9) has the trivial solution $\mathbf{B}_{\rm i} = 0$, because $\mathbf{B} \equiv \mathbf{B}_{\rm s}$ by definition. Now if we want to introduce a potential such as the standard vector potential $\mathbf{A}$, we can apply the Helmholtz theorem to this potential in the form $\mathbf{A} = \mathbf{A}_{\rm i} + \mathbf{A}_{\rm s}$, because in this way we can solve system (10) almost automatically by just putting

$$\mathbf{B}_{\rm s} = \nabla\times\mathbf{A}_{\rm s} \quad \text{and} \quad \mathbf{E}_{\rm s} = -\frac{1}{c}\frac{\partial\mathbf{A}_{\rm s}}{\partial t}. \quad (20)$$

Substituting (20) into the second equation of (10), we obtain

$$\nabla^2\mathbf{A}_{\rm s} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_{\rm s}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}_{\rm s}. \quad (21)$$

This equation with the corresponding boundary and initial conditions has the unique solution too. But probably this procedure is misleading due to its close resemblance to the conventional way of introducing the potentials. It is our claim that only with the use of the Helmholtz theorem and Maxwell's equations can we determine, totally, without any additional gauge condition, the electric and magnetic fields. So, after the use of the Helmholtz theorem, only Maxwell's equations are strong enough to determine the irrotational and solenoidal field components. One way to do so has been outlined with equations (20) and (21), but these equations and $\mathbf{A}$ are not necessary, because directly from (10) we can deduce a pair of inhomogeneous D'Alembert equations for $\mathbf{E}_{\rm s}$ and $\mathbf{B}_{\rm s}$. This pair of equations and corresponding boundary conditions determine the solenoidal fields; so, we have shown that with the Helmholtz theorem and Maxwell's equations, without additional gauge conditions, it is possible to obtain the electromagnetic field. However, we can use the solenoidal vector potential as a useful way to solve Maxwell's equations; in the later sections we shall see that its gauge invariance is a characteristic that leads us to think that it is a physically relevant vector field.

Thus, Maxwell's equations (1) after applying the Helmholtz theorem come down to the system of two equations (12) and (21), separated with respect to the vector and scalar potentials:

$$\left.\begin{aligned} \nabla^2\Phi &= -4\pi\varrho \\ \nabla^2\mathbf{A}_{\rm s} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_{\rm s}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{j}_{\rm s} \end{aligned}\right\}. \quad (22)$$

The most important characteristic of the potentials $\mathbf{A}_{\rm s}$ and Φ is that they, in common with the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields, are *invariant under the gauge transformations* (4), where $\mathbf{A}$ and φ are the generally accepted vector and scalar potentials of the electromagnetic field in an arbitrary gauge (or to be more exact, without any gauge).

Indeed, let us apply the Helmholtz theorem to the electromagnetic fields (2) expressed in terms of $\mathbf{A}$ and φ without taking into account any gauge condition. Applying the theorem to (2), we have

$$\mathbf{E}_{\rm i} = -\nabla\varphi - \frac{1}{c}\frac{\partial\mathbf{A}_{\rm i}}{\partial t}, \quad (23)$$

$$\mathbf{E}_{\rm s} = -\frac{1}{c}\frac{\partial\mathbf{A}_{\rm s}}{\partial t}, \quad (24)$$

$$\mathbf{B}_{\rm i} = 0, \quad (25)$$

$$\mathbf{B}_{\rm s} = \nabla\times\mathbf{A}_{\rm s}. \quad (26)$$

One can represent the potential $\mathbf{A}_i$, by definition, as

$$\mathbf{A}_{\rm i} = -\nabla\varphi_A, \quad (27)$$

where φ_A is some scalar function. Substituting equation (27) into equation (23), we obtain

$$\mathbf{E}_{\rm i} = -\nabla\varphi + \frac{1}{c}\nabla\frac{\partial\varphi_A}{\partial t} = -\nabla\left(\varphi - \frac{1}{c}\frac{\partial\varphi_A}{\partial t}\right). \quad (28)$$

From equations (28) and (11), one can see that the relation of Φ to φ is

$$\Phi = \varphi - \frac{1}{c}\frac{\partial\varphi_A}{\partial t}. \quad (29)$$

Now, if we apply the gauge transformations (4) and the Helmholtz theorem, from

$$\mathbf{A}' = \mathbf{A}'_{\rm i} + \mathbf{A}'_{\rm s} = \mathbf{A}_{\rm i} + \mathbf{A}_{\rm s} + \nabla\psi, \quad (30)$$

equating solenoidal parts, we obtain

$$\mathbf{A}'_{\rm s} = \mathbf{A}_{\rm s}. \quad (31)$$

In order to obtain the transformation law for Φ (equation (29)), we at first find the corresponding transformation for φ_A. From equation (30), we obtain for the irrotational part of $\mathbf{A}'$

$$\mathbf{A}'_{\rm i} = \mathbf{A}_{\rm i} + \nabla\psi, \quad (32)$$

and substituting equation (27) into this equation, we have

$$-\nabla\varphi'_A = -\nabla\varphi_A + \nabla\psi = -\nabla(\varphi_A - \psi) \Rightarrow \varphi'_A = \varphi_A - \psi. \quad (33)$$

Finally, using equation (29) for Φ' and taking into account equations (4) and (33), we obtain

$$\Phi' = \varphi' - \frac{1}{c}\frac{\partial\varphi'_A}{\partial t} = \left(\varphi - \frac{1}{c}\frac{\partial\psi}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}(\varphi_A - \psi) = \varphi - \frac{1}{c}\frac{\partial\varphi_A}{\partial t} = \Phi. \quad (34)$$

Thus, we have ascertained that $\mathbf{A}_\mathrm{s}$ and Φ are *invariant under the gauge transformations*.

We have to note that the gauge transformations affect only the component $\mathbf{A}_\mathrm{i}$, but it does not participate in the definitions of the vectors $\mathbf{E}$ and $\mathbf{B}$ of the electromagnetic field:

$$\left.\begin{aligned} \mathbf{E} &= -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}_\mathrm{s}}{\partial t} \\ \mathbf{B} &= \nabla\times\mathbf{A}_\mathrm{s} \end{aligned}\right\}. \quad (35)$$

Perhaps this calculation represents a hard rationale for something that should be clear from the start: any gauge transformation is irrelevant if we use the Helmholtz theorem, and in this way, we eliminate any objection to our method based on the idea that with a gauge transformation we can find another representation for the fields so that the action at a distance can be eliminated, but this is not actually the case: within the scope of the Helmholtz theorem, action-at-a-distance fields cannot be eliminated using gauge transformations, because the basic field variables, the potentials, do not change with their help, and that change of representation is the whole point of gauge transformations.

3. The puzzle with gauge transformations

The Helmholtz theorem gives us a representation for the electromagnetic field that, jointly with Maxwell's equations, allows us to know the fields in a unique way once we define the boundary value problem for the four differential equations (22). Besides, using again the Helmholtz theorem we can show that the introduced potentials are gauge invariant, a result that implies that any theory of gauge transformations is useless because the whole point of gauge transformations is missed: we cannot use them to change the equations that satisfy the potentials and not even the potentials themselves. For this reason, the potentials introduced with the help of the Helmholtz theorem are not the same, and cannot be related to the usual potentials, which we may call 'g-potentials'. However, there is a drawback in the Helmholtz representation: it introduces a component $\mathbf{E}_\mathrm{i}$ which shows action at a distance. Obviously, this is in open conflict with special relativity, a point that has taken some people to the belief that this component must be a mathematically introduced function without any counterpart in physical reality that depends somehow on the Lorentz invariant functions or on solutions to Lorentz invariant equations [7]. We have already discussed this objection [8] (see also [9]), showing that $\mathbf{E}_\mathrm{i}$ is a vector function independent of $\mathbf{E}_\mathrm{s}$; hence, at least this objection cannot be considered seriously. Anyway the main objection remains: what about Lorentz invariance? At this point, we must be careful, because the electric and magnetic vector fields are not Lorentz invariant, but the equations that define them are. These fields are related to their counterparts in any inertial frame by means of a Lorentz transformation, and this is the meaning of the Lorentz invariance of the D'Alembert equations: if we have a given solution on one inertial frame, on any other frame we can build a solution using a Lorentz transformation without in any way altering the functional form of the field equations. It is clear that an isolated electric field without magnetic counterpart is not related to another field on another reference frame by a Lorentz transformation [10] and this is not a reason for rejecting it, because in the low velocity limit the right transformation group is the Galileo group. Therefore, our best explanation for the presence of action-at-a-distance solutions to Maxwell's equations is as follows: imagine that the Lorentz invariance of the D'Alembert equation allows us to define a group orbit starting from a given initial solution; in this way starting with a set of solutions to the D'Alembert equation we can build, with the help of Lorentz transformations, any other solution. However, when we use the Helmholtz theorem an isolated electric field is present, such that its field equation is not Lorentz invariant (or covariant); hence, we cannot build its solutions using Lorentz transformations, obtaining a completely nonrelativistic field that is, however, a necessary piece for the solution that we obtain for Maxwell's equations. Hence, this solution is not the set of usual solutions that can be obtained with the help of special relativity or gauge transformation theory. The question is whether this explanation is sound at all or whether it is only an ad hoc rationale for non-physical solutions that can be rejected using special relativity postulates. Let us try to show that this is not the case and that even in the usual framework of gauge transformations, there arise action-at-a-distance solutions that, on the basis of general ideas, cannot be eliminated; hence, our challenge here is going to be directed against gauge transformation theory, not special relativity.

It is clear that an isolated electric field without a magnetic counterpart is not a Lorentz invariant concept [10] and this is not a reason for rejecting it, because in the low velocity limit, no function is a Lorentz invariant. Our best explanation is to assert that the Helmholtz representation selects functions in the space of solutions to Maxwell's equations, which are not Lorentz invariant. If we suppose that Lorentz symmetry relates two solutions in the space of solutions to Maxwell's equations, defining an orbit, the Helmholtz theorem shows that not all solutions of Maxwell's equations lie on the orbits of the Lorentz group. This could be considered as a weak reason to believe in the existence of non-Lorentz invariant solutions to Maxwell's equations; hence, let us show that even in the conventional approach these kinds of solutions are possible.

The usual g-potentials present these types of solutions too when the Coulomb gauge is used, but these solutions are considered harmless because there exist gauge transformations that relate potentials that satisfy a non-Lorentz invariant equation with potentials that satisfy a Lorentz invariant one. This answer is complemented with the hope that such a transformation is easy to obtain or, to say the least, that its existence can be proved in principle. This possibility is implicit in the framework that needs gauge transformations, because in this framework the potentials are non-gauge invariant and can be changed at will, or that is the hope. We are not aware of such a general existence proof, but now we shall consider some general ideas that lead us to believe that if such a proof exists, it is quite restrictive. Therefore, if the method proposed in the previous section is rejected because it leads to action-at-a-distance solutions, the standard method in the Coulomb gauge must be rejected for the same reason. Let us suppose that a gauge transformation relating g-potentials exists in the Coulomb gauge $\langle \mathbf{A}_C, \varphi_C \rangle$ with g-potentials in the Lorentz gauge $\langle \mathbf{A}_L, \varphi_L \rangle$. Hence, we can write down the equations

$$\left.\begin{aligned} \mathbf{A}_C &= \mathbf{A}_L + \nabla\psi \\ \varphi_C &= \varphi_L - \frac{1}{c}\frac{\partial\psi}{\partial t} \end{aligned}\right\}. \tag{36}$$

But with the advanced supposition we have made *petitio principii*, by following this way we prove nothing because we have introduced the function whose existence we want to prove. The right procedure is to show that the differential form

$$(\mathbf{A}_C - \mathbf{A}_L)\cdot \mathrm{d}\mathbf{r} + c(\varphi_C - \varphi_L)\,\mathrm{d}t \tag{37}$$

is integrable.

There are two kinds of integrability concepts for this differential form: the local one that makes use of cross differentiation and the global one that makes use of a procedure of integration along paths in the space of definition of the one-form. Local integrability gives us a set of differential equations that must be identically satisfied by the g-potentials, and must be used case by case. Global integrability requires the consideration of space connectivity; that is, if the space in which we define the one-form is simply connected, a gauge function globally defined exists, whereas otherwise it does not exist, and on each connected component of the space, a different function may exist or not and a matching procedure can be employed or not. Now, directly from the gauge transformation and the gauge conditions, it is very easy to deduce that the gauge function ψ must satisfy the following coupled pair of differential equations:

$$\Delta\psi = -\frac{1}{c}\frac{\partial\varphi_L}{\partial t}, \quad \nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\frac{1}{c}\frac{\partial\varphi_C}{\partial t}. \tag{38}$$

Hence, if the one-form is integrable this pair of differential equations is satisfied, and the integrability conditions of the one-form turn out to be the integrability conditions of these equations. Therefore, it is non-trivial to prove the existence of the gauge function that transforms the g-potentials in the Coulomb gauge into g-potentials in the Lorenz gauge, and in this sense such a transformation is not generally guaranteed and the certainty that we can eliminate the action-at-a-distance fields is lost even in the standard approach. Obviously we can support our belief in the standard approach for many reasons, for example a blind faith in special relativity theory, but we cannot be confident that we can eliminate the action-at-a-distance fields in all cases.

4. Conclusions

Now we point out the following characteristics of the potentials $\mathbf{A}_s$ and Φ, which are solutions of equations (22),

$$\left.\begin{aligned} \nabla^2\Phi &= 4\pi\varrho, \\ \nabla^2\mathbf{A}_s - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_s}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{j}_s \end{aligned}\right\}:$$

1. These equations are already separated with respect to vector and scalar potentials, so there is *no necessity* of using the gauge transformations and, accordingly, of making use of either the Lorentz or Coulomb condition (unlike the case of the *conventional potentials* $\mathbf{A}$ and φ (see equations (3)), when one must introduce either the Lorentz or Coulomb condition (5) in order to separate equations for these potentials. Note again that our approach saves us the trouble of using both the Lorentz and Coulomb conditions (see e.g. [1, 2]).
2. The vector potential $\mathbf{A}_s$ (equation (21)) and the scalar potential Φ (equation (12)) introduced thus are uniquely defined. The scalar potential Φ is a generator of the so-called instantaneous action at a distance, whereas the solenoidal vector potential $\mathbf{A}_s$ can propagate with the velocity of light and it is responsible for the retarded action of the electromagnetic field (obviously, equation (21) can have non-wave solutions, too).
3. Because of their one-valued definiteness and for the reason that the potentials $\mathbf{A}_s$ and Φ are *invariant under the gauge transformations*, the potentials $\mathbf{A}_s$ and Φ are, by their nature, *physical quantities* (unlike the *conventional potentials* $\mathbf{A}$ and φ, which are regarded as nothing more than an auxiliary mathematical tool, which does not reflect a physically existing phenomenon) and completely characterize the electromagnetic field.
4. Our criterion to consider a mathematical function as a measurable physical quantity is that this function remains invariant under some group action (in our case, the action of the gauge group); in addition, it must be well defined by the mathematical problem; that is, it must exist in the mathematical sense. These are necessary conditions of physical reality borrowed from the properties of the electromagnetic field, because the electric and magnetic fields are gauge invariants and in most or in all the problems are well defined by the boundary value problem for the field equations. Therefore, by analogy, we suppose that they are sufficient conditions of physical reality. Hence, in fact, we can consider this supposition to be a prediction of our theoretical method of solution for Maxwell's equations.
5. The physical meaning of $\mathbf{A}_s$ in classical electrodynamics is that its variation in time and in space generates the solenoidal fields $\mathbf{E}_s$ and $\mathbf{B}_s$ (equations (24) and (26)). On the other hand, the spatial variation in Φ gives the irrotational field $\mathbf{E}_i$ (equation (11)). Hence, one can conclude from this that the electromagnetic potentials $\mathbf{A}_s$ and Φ are real *physical quantities* in addition to the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields. In corroboration of

this conclusion, we quote from the book by R Feynman *et al* [11] (section 15–5): '*… is the vector potential a 'real' field? …for a long time it was believed that* **A** *was not a 'real' field. … there are phenomena involving quantum mechanics which show that in fact* **A** *is a 'real' field in the sense that we have defined it …* **E** *and* **B** *are slowly disappearing from the modern expression of physical laws; they are being replaced by* **A** [the vector potential] *and* φ [the scalar potential]'.

6. Note that the Aharonov–Bohm effect proves the physical reality of the vector potential $\mathbf{A}_s$. Indeed, in their original work [12], Aharonov and Bohm showed that the phase shift of the electron wave function ought to be

$$\Delta S/\hbar = -\frac{e}{c\hbar}\oint \mathbf{A}\cdot d\mathbf{x}, \tag{39}$$

where $\oint \mathbf{A}\cdot d\mathbf{x} = \int \mathbf{B}\cdot d\mathbf{s}$ is the total magnetic flux inside the circuit (see [12], p 486). But in our approach $\nabla\times\mathbf{A} = \nabla\times\mathbf{A}_s + \nabla\times\mathbf{A}_i$ and $\nabla\times\mathbf{A}_i = 0$ by definition; hence by the Stokes theorem

$$\Delta S/\hbar = -\frac{e}{c\hbar}\oint \mathbf{A}_s\cdot d\mathbf{x}. \tag{40}$$

7. The physical reality of the scalar electric potential was demonstrated in 1998 by van Oudenaarden *et al* [13], who showed that just as the phase shift of the wave function depends on the magnetic vector potential, it also depends on the scalar electric potential. It is obvious that the electric potential Φ plays a leading role in this effect [13], because in the static case our Φ coincides with the conventional φ.

Hence, we have proved that one can introduce, uniquely defined, electromagnetic potentials $\mathbf{A}_s$ and Φ, which completely characterize the electromagnetic field, and that there is *no necessity to introduce the gauge transformations.* Thus, we have ascertained that the quantum-mechanical *reality* (see e.g. [11–14]) of the electromagnetic potentials is also a *classical-electrodynamics fact* (note that the problem of the *physical reality* of electromagnetic potentials within the framework of the classical-electrodynamics theory has already been discussed in the literature; see e.g. [15–19]). Thus, we have reason to expect that in future, classical-electrodynamics experiments, which confirm the physical reality of electromagnetic potentials $\mathbf{A}_s$ and Φ, could be performed. It must be emphasized that Φ and $\mathbf{A}_s$ naturally emerge from our method without artificially including a gauge condition.

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