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Spherical and Ring-Like Configurations for the Gravitodynamical Field

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ABSTRACT

In the present work, we write a brief exposition of the Jefimenko's theory of gravitation. This theory arose from the analogy between the laws of gravitation and electromagnetism, that is, exist a second gravitational field called cogravitational field, analogous to the magnetic field. We introduce a new system of units called Gravitational Gaussian System (GGS). This system allows us write the equations of gravitation in a simple form to solve them. Using the Jefimenko's equations of gravitation, we obtain the wave equations for gravitational and cogravitational fields and we find wave solutions. We demonstrate there are configurations of the gravitodynamical field (that is, the set of gravitational and cogravitacional fields) in form of co gravitational field spheres and gravitational field rings. This phenomenon must be an analogue of the ball lightning in the electromagnetic field, but in this case the cogravitational field spheres serves as containers of matter (it could be a gas). We analyze how this configuration acts on the particles inside the spheres, and we investigate the physical properties of such configurations, namely, how behaves the density of energy and the Pointing vector of this solution.

Keywords: *Cogravitation; Gravitational waves; Gravitodynamical field.*

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INTRODUCTION

The Jefimenko's equations arise from an analogy between the laws that rules the electrodynamics and the laws of gravitation. Such analogy was proposed for the first time by Heaviside in a paper published [1] more than a century ago, where he supposed there must exist a second field due to moving masses and acting over moving masses only, called by Jefimenko, cogravitational field (sometimes this field is called Heaviside's field). The Heaviside paper was forgotten for a long time until Jefimenko returned his work and made improvement to the Heaviside's work in two books published and reissued since the 90's decade [2], [3].

To start we need to write about the analogy made by Jefimenko between the laws of electromagnetism and the laws of gravitation. Oliver Heaviside proposed a system of equations with the same structure as the Maxwell's equations, assuming the existence of a second field due and acting over moving masses only, called Heaviside's field or co gravitational field and it is denoted by **k**, unlike the ordinary gravitational field **g** which is due and acts not only on stationary masses but in movement. Although there are detractors of the Jefimenko's theory of gravitation[1] (see for example [5]), there exist books written by Wenceslao Segura [6] and Jolien E. D. Creighton [7] where they derived the Jefimenko's equations from the linearized Einstein's equations. Jefimenko derived the equations of gravitodynamics in a different way, starting from make an analogy of his retarded solutions of the electromagnetic field to the gravitodynamical field to finally get the analogous equations to the Maxwell's equations for the gravitational and cogravitational fields.

Next, we assume there is a configuration of the gravitodynamical field in an analogue way to those found in other works realized by Chubykalo and Espinoza [8], [9], where the authors have obtained the mathematical foundations on the Kapitsa's hypothesis [10] about the origin of ball lightning related with interference processes. The configuration of the gravitodynamical field in our work begins with

[1]Called by us in a previous work *gravitodynamical theory* [4].

the hypothesis that the gravitational waves exist. We are going to do a formal development about this issue based in the theoretical results obtained by Jefimenko. We will use the free gravitodynamical equations, that is, the equations valid for vacuum.

It is important to emphasize that gravitational waves were predicted in the general theory of relativity by linearzing Einstein's field equations and this approximation is valid for weak fields. It is obvious that a complete theory of gravity is not linear but this linear approximation allows us to study a great variety of gravitational phenomena where gravitational induction is considered. In the same way that electromagnetic theory is not linear but Maxwell equations are applicable to a wide range of electromagnetic phenomena.

Our results presented here do not try to replace the non-linear theories of gravitation, such as Einstein's theory of general relativity [11] or Logunov's relativistic theory of gravitation [12], but we want to show the importance of a linear gravitational theory, not only historical but methodological too, because we will show that exist properties of the weak fields that are not understood because they were not have the opportunity to appear in a linear theory of gravitation.

THE JEFIMENKO'S EQUATIONS FOR GRAVITATION

The gravitodynamical theory is a generalization of the Newton's gravitation theory, since the Newtonian theory of gravitation describes perfectly phenomena of a wide range of masses, but it does not consider the behavior of the fields of moving mass distributions. This is the reason why it is necessary to make an extension of the classical Newton's theory of gravitation.

We can write the Newton's theory in a way as a force field theory in terms of the gravitational field **g** as

$$\nabla\cdot\mathbf{g} = -4\pi\gamma\varrho \tag{1}$$

and

$$\nabla\times\mathbf{g} = 0, \tag{2}$$

where γ is the gravitational constant, ϱ is the mass density given by $\varrho = dm/dV$ and dm is the element of mass contained in the volume element dV.

As we know, the gravitational field **g** is defined by means of the force exerted by the mass distribution ϱ over a mass test

$$\mathbf{F}_g = m_t\mathbf{g}. \tag{3}$$

In other words, the gravitational field is the perturbation of the space due to distribution of mass in some region which interact on a test mass m_t Both masses (the mass creating the field and the test mass) can be moving or at rest.

If we consider that the co gravitational field **k** exists, we need to define it in terms of the co gravitational force

$$\mathbf{F}_k = m_t(\mathbf{v}\times\mathbf{k}), \tag{4}$$

Where $\mathbf{F}_k$ is the force exerted by the cogravitational field over a test mass m_t moving with velocity **v**.

So, we can define the cogravitational field **k** as the perturbation of the space due to a moving mass distribution which interacts on a moving test mass.

Jefimenko started to derive their gravitational equations from the next expressions

$$\mathbf{g} = -\gamma\int\left\{\frac{[\varrho]}{r^3}+\frac{1}{r^2c}\left[\frac{\partial\varrho}{\partial t}\right]\right\}\mathbf{r}dV' + \frac{\gamma}{c^2}\int\frac{1}{r}\left[\frac{\partial(\varrho\mathbf{v})}{\partial t}\right]dV' \tag{5}$$

and

$$\mathbf{k} = -\frac{\gamma}{c^2}\int\left\{\frac{[\varrho\mathbf{v}]}{r^3}+\frac{1}{r^2c}\left[\frac{\partial(\varrho\mathbf{v})}{\partial t}\right]\right\}\times\mathbf{r}dV', \tag{6}$$

where γ is the gravitational constant, c is the velocity of propagation of the fields[2] and **r** is the vector directed from the element of volume dV' (the source point) to the point where the gravitodynamical field is measured (the field point) and r is its magnitude, the square brackets designate that the quantities inside them are evaluated in the delayed time$t' = t - r/c$. The integrals are evaluated over all space.

We can see from (5) and (6) that the gravitodynamical fields have four causative sources, namely: the mass density ϱ, the temporal derivative of the mass density $\partial_t\varrho$, the mass current ϱ**v**and its time derivative $\partial_t(\varrho\mathbf{v})$.

Jefimenko obtained the gravitodynamical equations making use of the vector calculus and some vector identities. The gravitodynamical equations are:

$$\nabla\cdot\mathbf{g} = -4\pi\gamma\varrho, \tag{7}$$

$$\nabla\cdot\mathbf{k} = 0, \tag{8}$$

$$\nabla\times\mathbf{g} = -\frac{\partial\mathbf{k}}{\partial t}, \tag{9}$$

[2]Jefimenko assumed that the velocity of the propagation of the fields must be c, i.e, the finite speed of light. But we have demonstrated in [4] that this velocity must be finite or instantaneous.

where $\mathbf{j} = \varrho\mathbf{v}$ is the mass current density. It is evident the analogy between the Maxwell equations and the Jefimenko's ones is not perfect: while we have only one kind of mass, we have two types of electric charge. While the electric field is directed from the positive charges that generate it and is directed toward negative charges, the gravitational field is always directed to the masses by which it is created. Another difference is that the magnetic field is dextrorotatory (right hand) with respect to the electric current through which it is generated, while the gravitational field is always levorotatory (left hand) with respect to the mass current through which it is generated. In spite of these differences, the system of equations obtained by Jefimenko describe correctly the behavior of the weak gravitational fields, as we have already seen they are deduced from different formulations in [3] and in [6], [7].

We define a Gaussian system of units for the gravitodynamical field, in order to simplify our calculations. We will call this system Gravitational Gaussian System (GGS).To do this we need to introduce the next rationalized quantities:

Table1. *Rationalized quantities in the new gravitational Gaussian system of units*[3].

	Gravitational field	Co gravitational field
Formula	$\mathbf{G} = \gamma^{-1}\mathbf{g}$	$\mathbf{K} = \gamma^{-1}c\mathbf{k}$
Units	$[\mathbf{G}] = ML^{-2}$ =*Jef*	$[\mathbf{K}] = ML^{-2}$ =*Jef*

If we introduce such quantities in the system of Equations (7)-(10) we obtain the next system of equations in the GGS

$$\nabla \cdot \mathbf{G} = -4\pi\varrho, \quad (11)$$

$$\nabla \cdot \mathbf{K} = 0, \quad (12)$$

$$\nabla \times \mathbf{G} = -\frac{1}{c}\frac{\partial \mathbf{K}}{\partial t}, \quad (13)$$

$$\nabla \times \mathbf{K} - \frac{1}{c}\frac{\partial \mathbf{G}}{\partial t} = -\frac{4\pi}{c}\mathbf{j}. \quad (14)$$

Where, $\mathbf{j} = \varrho\mathbf{v}$is the mass current density and $\mathbf{v}$is the velocity of the mass distribution generating the co gravitational field.

Gravitational Waves

The Jefimenko's theory of gravitation predicts also the existence of gravitational waves. We

$$\nabla \times \mathbf{k} - \frac{1}{c^2}\frac{\partial \mathbf{g}}{\partial t} = -\frac{4\pi\gamma}{c^2}\mathbf{j}, \quad (10)$$

will study the properties of such waves, in this and in other sections (especially in the section V, where we will study the energy and the Poynting vector of such waves).

We will obtain in this section the wave equation for both fields, namely, the gravitational field and the cogravitational one, making direct calculations on the system of equations (11)-(14), we will see that these equations lead us to the wave equation. We start calculating the curl on the equation (13)

$$\nabla \times \nabla \times \mathbf{G} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \mathbf{K}, \quad (15)$$

and substituting in equation (14) and using the identity for a Laplacian of a vector,

$$\nabla \times \nabla \times \mathbf{V} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{V}) - \Delta\mathbf{V}, \quad (16)$$

for any arbitrary vector $\mathbf{V}$, and where $\Delta = \nabla^2$is the Laplacian operator, we get

$$\Delta\mathbf{G} - \frac{1}{c^2}\frac{\partial^2 \mathbf{G}}{\partial t^2} = -4\pi\left(\nabla\varrho + \frac{1}{c^2}\frac{\partial \mathbf{j}}{\partial t}\right), \quad (17)$$

the in homogeneous gravitational wave equation.

In a similar way, starting from (14) taking the curl and using the identity (16), we get

$$\Delta\mathbf{K} - \frac{1}{c^2}\frac{\partial^2 \mathbf{K}}{\partial t^2} = \frac{4\pi}{c}\nabla \times \mathbf{j}, \quad (18)$$

Is the inhomogeneous cogravitational wave equation. Both expressions (17) and (18) are field waves propagating on the space with a velocity c.

If we consider regions without masses distributions and current masses we obtain the homogeneous wave equations, namely,

$$\Delta\mathbf{G} - \frac{1}{c^2}\frac{\partial^2 \mathbf{G}}{\partial t^2} = 0 \quad (19)$$

And

$$\Delta\mathbf{K} - \frac{1}{c^2}\frac{\partial^2 \mathbf{K}}{\partial t^2} = 0. \quad (20)$$

The equations obtained (19) and (20) can be solved by a sum of two vector functions, $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$,

$$\boldsymbol{\varphi}_1(\boldsymbol{k} \cdot \mathbf{r} - \omega t) + \boldsymbol{\varphi}_2(\boldsymbol{k} \cdot \mathbf{r} + \omega t), \quad (21)$$

Where $\boldsymbol{k} = (k_x, k_y, k_z)$ is the wave vector[4], $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are general expressions which represent plane waves propagating with velocity c in

[3]We define the unit *Jefimenko* abbreviated *Jef* for the rationalized gravitational and co gravitational fields in honor of Jefimenko.

[4] To avoid confusions, we use italic boldface $\boldsymbol{k}$ to represent the wave vector and normal boldface $\mathbf{k}$ to represent the cogravitational field.

opposite directions. Then the solution (21) for the wave equation can be derived from the method of separation of variables. If we introduce the harmonic dependence given by

$$\mathbf{G} = G_0 e^{i(\boldsymbol{k}\cdot\mathbf{r}\pm\omega t)}\boldsymbol{n}_{\mathbf{G}} \quad (22)$$

and

$$\mathrm{K} = K_0 e^{i(\boldsymbol{k}\cdot\mathbf{r}\pm\omega t)}\boldsymbol{n}_{\mathbf{K}}, \quad (23)$$

Where $\boldsymbol{n}_{\mathbf{G}}$ is a unitary vector in the direction of propagation of $\mathbf{G}$ and $\boldsymbol{n}_{\mathrm{K}}$ is in the direction of propagation of $\mathbf{K}$, the wave's equation results in the dispersion's relation

$$\omega^2 - k^2c^2 = 0. \quad (24)$$

For the election of sign $k = +\omega/c$, we obtain from the Jefimenko's equations for gravitodynamical fields

$$\boldsymbol{k}\cdot\boldsymbol{n}_{\mathbf{G}} = 0 \text{ and } \boldsymbol{g}\cdot\boldsymbol{n}_{\mathrm{K}} = 0, \quad (25)$$

$$(\boldsymbol{k}\times\boldsymbol{n}_{\mathbf{G}})G_0 = \frac{\omega}{c}K_0\boldsymbol{n}_{\mathbf{K}} \text{ and } (\boldsymbol{k}\times\boldsymbol{n}_{\mathrm{K}})K_0 = \frac{\omega}{c}G_0\boldsymbol{n}_{\mathbf{G}}. \quad (26)$$

From Eq s. (25) And (26) we can see that $\mathbf{G}$ and $\mathbf{K}$ are vector mutually perpendicular to the direction of propagation and$|G_0| = |K_0|$.

An Interesting Wave Solution of the Jefimenko's Equations for the Free Space

The prediction of gravitational waves by means of the Jefimenko's gravitodynamical theory let us search a various types of solutions for the gravitodynamical field in vacuum. For example, as we will show in this section, solutions exist that have a set of interesting properties. These solutions have the form of spheres of co gravitational field and ring-like gravitational field, in such way the total configuration of the gravitodynamical field oscillates. In such spheres the Heaviside's field is tangent in all points over the surface of this sphere, and the same for the ring-like configuration of the gravitational field.

We will begin this section rewriting the Jefimenko's equations for the gravitation, assuming there are regions of free space or vacuum, this means, $\varrho = 0$ and $\mathbf{j} = 0$. We get

$$\nabla\cdot\mathbf{G} = 0, \quad (27)$$

$$\nabla\cdot\mathbf{K} = 0, \quad (28)$$

$$\nabla\times\mathbf{G} = -\frac{1}{c}\frac{\partial\mathbf{K}}{\partial t}, \quad (29)$$

$$\nabla\times\mathbf{K} = \frac{1}{c}\frac{\partial\mathbf{G}}{\partial t}. \quad (30)$$

In Electrodynamics is usual to refer to standard polarity in the solutions of the Maxwell equations when the electric field $\mathbf{E}$ is a polar vector and the magnetic induction $\mathbf{B}$ is an axial vector, or pseudo-vector. This means that after a transformation of inversion of axis coordinates, $\mathbf{E}$ changes its signs, while $\mathbf{B}$ maintains its signs. Following the analogy between both theories we are going to look for solutions to these equations with standard polarity, to wit, when the vector $\mathbf{G}$ is polarand $\mathbf{K}$ is axial. We propose to solve the system of free Jefimenko's equations by the method of separation of variables, we can write $\mathbf{G}$ and $\mathbf{K}$ as follows:

$$\mathbf{G}(\mathbf{r},t) = \boldsymbol{\gamma}(\mathbf{r})\mu(t) \quad (31)$$

and

$$\mathbf{K}(\mathbf{r},t) = \boldsymbol{\kappa}(\mathbf{r})\nu(t), \quad (32)$$

where $\boldsymbol{\gamma}(\mathbf{r})$ is a polar vector and $\boldsymbol{\kappa}(\mathbf{r})$ is an axial one, also $\mu(t)$ and $\nu(t)$ are functions of time.

Substituting (31) and (32) in the system (27)-(30)

$$\nabla\cdot\boldsymbol{\gamma}(\mathbf{r}) = 0, \quad (33)$$

$$\nabla\cdot\boldsymbol{\kappa}(\mathbf{r}) = 0, \quad (34)$$

$$\nabla\times\boldsymbol{\gamma}(\mathbf{r}) = -\frac{1}{c}\frac{1}{\mu(t)}\frac{\partial\nu(t)}{\partial t}\boldsymbol{\kappa}(\mathbf{r}), \quad (35)$$

$$\nabla\times\boldsymbol{\kappa}(\mathbf{r}) = \frac{1}{c}\frac{1}{\nu(t)}\frac{\partial\mu(t)}{\partial t}\boldsymbol{\gamma}(\mathbf{r}). \quad (36)$$

We can equate the temporal parts of both equations to certain constants to obtain a consistent system,

$$-\frac{1}{\mu(t)}\frac{\partial\nu(t)}{\partial t} = \omega_1 \quad (37)$$

And

$$\frac{1}{\nu(t)}\frac{\partial\mu(t)}{\partial t} = \omega_2. \quad (38)$$

We are going to equating both equations to ω, in order to obtain sinusoidal solutions and get only three constants in our system (37)-(38)

$$\nu(t) = A\cos(\omega t - \delta) \quad (39)$$

and

$$\mu(t) = A\sin(\omega t - \delta), \quad (40)$$

where, A and δ are arbitrary constants.

Thus, in this way, the equations for $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ become

$$\nabla \times \boldsymbol{\gamma}(\mathbf{r}) = \frac{\omega}{c}\boldsymbol{\kappa}(\mathbf{r}), \tag{41}$$

and

$$\nabla \times \boldsymbol{\kappa}(\mathbf{r}) = \frac{\omega}{c}\boldsymbol{\gamma}(\mathbf{r}). \tag{42}$$

Due to the linearity of the spatial components of the fields, we can add (41) and (42), then, we can define the vector

$$\boldsymbol{\sigma}(\mathbf{r}) = \boldsymbol{\gamma}(\mathbf{r}) + \boldsymbol{\kappa}(\mathbf{r}), \tag{43}$$

Such as, we obtain

$$\nabla \times \boldsymbol{\sigma}(\mathbf{r}) = \frac{\omega}{c}\boldsymbol{\sigma}(\mathbf{r}). \tag{44}$$

First of all, we are going to solve (44), then, once we have obtained $\boldsymbol{\sigma}$, we can calculate $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$. We need to note that vector $\boldsymbol{\sigma}$ has not polarity, but we can express its polar and axial parts as

$$\boldsymbol{\gamma}(\mathbf{r}) = \frac{1}{2}\{\boldsymbol{\sigma}(\mathbf{r}) - \boldsymbol{\sigma}(-\mathbf{r})\} \tag{45}$$

And

$$\boldsymbol{\kappa}(\mathbf{r}) = \frac{1}{2}\{\boldsymbol{\sigma}(\mathbf{r}) + \boldsymbol{\sigma}(-\mathbf{r})\}. \tag{46}$$

Taking the curl of (45) and (46), and inverting the coordinates of equation (44), we have

$$\nabla \times \boldsymbol{\gamma}(\mathbf{r}) = \frac{1}{2}\{\nabla \times \boldsymbol{\sigma}(\mathbf{r}) - \nabla \times \boldsymbol{\sigma}(-\mathbf{r})\}$$
$$= \frac{1}{2}\left\{\frac{\omega}{c}\boldsymbol{\sigma}(\mathbf{r}) + \frac{\omega}{c}\boldsymbol{\sigma}(-\mathbf{r})\right\}$$
$$= \frac{\omega}{c}\boldsymbol{\kappa}(\mathbf{r}) \tag{47}$$

And

$$\nabla \times \boldsymbol{\kappa}(\mathbf{r}) = \frac{1}{2}\{\nabla \times \boldsymbol{\sigma}(\mathbf{r}) + \nabla \times \boldsymbol{\sigma}(-\mathbf{r})\} = \frac{1}{2}\left\{\frac{\omega}{c}\boldsymbol{\sigma}(\mathbf{r}) - \frac{\omega}{c}\boldsymbol{\sigma}(-\mathbf{r})\right\} = \frac{\omega}{c}\boldsymbol{\gamma}(\mathbf{r}), \tag{48}$$

we can be sure that the system is satisfied. The only thing we need to do is to find the solution of (44), in order to find the solution of the system (41)-(42). To get such solution we will consider the vector $\boldsymbol{\sigma}$ in spherical coordinates and we suppose that the solution has axial symmetry

$$\boldsymbol{\sigma} = \sigma_r(r,\theta)\hat{\boldsymbol{r}} + \sigma_\theta(r,\theta)\hat{\boldsymbol{\theta}} + \sigma_\phi(r,\theta)\hat{\boldsymbol{\varphi}}. \tag{49}$$

The curl of $\boldsymbol{\sigma}$ in spherical coordinates is

$$\nabla \times \boldsymbol{\sigma} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial(r\sigma_\phi\sin\theta)}{\partial\theta} - \frac{\partial(r\sigma_\theta)}{\partial\phi}\right\}\hat{\boldsymbol{r}} + \frac{1}{r\sin\theta}\left\{\frac{\partial(\sigma_r)}{\partial\phi} - \frac{\partial(r\sigma_\phi\sin\theta)}{\partial r}\right\}\hat{\boldsymbol{\theta}} + \frac{1}{r}\left\{\frac{\partial(r\sigma_\theta)}{\partial r} - \frac{\partial(\sigma_r)}{\partial\theta}\right\}\hat{\boldsymbol{\varphi}}, \tag{50}$$

We can obtain the next system of equations taking into account Eq. (44) and comparing it with (49) and (50)

$$\frac{\partial(\sigma_\phi\sin\theta)}{\partial\theta} = \frac{\omega r\sigma_r\sin\theta}{c}, \tag{51}$$

$$\frac{\partial(r\sigma_\phi)}{\partial r} = -\frac{\omega r\sigma_\theta}{c} \tag{52}$$

and

$$\frac{\partial(r\sigma_\theta)}{\partial r} - \frac{\partial(\sigma_r)}{\partial\theta} = \frac{\omega r\sigma_\phi}{c}. \tag{53}$$

Expressing the variables σ_r y σ_θ from (51) y (52) and replacing them in (53), we obtain the next partial differential equation for σ_ϕ, namely,

$$r\frac{\partial^2}{\partial r^2}(r\sigma_\phi) + \frac{\partial}{\partial\theta}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sigma_\phi\sin\theta)\right\} + \frac{\omega^2r^2}{c^2}\sigma_\phi = 0. \tag{54}$$

If we propose $\sigma_\phi = R(r)\Theta(\theta)$ as a solution for (54) we find that R and Θ have to satisfy

$$r^2\frac{d^2(rR)}{dr^2} + \left(\frac{\omega^2r^2}{c^2} + \lambda\right)rR = 0 \tag{55}$$

And

$$\frac{d}{d\theta}\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}(\Theta\sin\theta)\right\} - \lambda\Theta = 0, \tag{56}$$

where λ is an arbitrary constant? If $\lambda = 0$, then the solution for rR in Equation (55) must be $A\cos\frac{\omega r}{c} + B\sin\frac{\omega r}{c}$, where A and B are constants, but, in general, A and B depend on r, so

$$rR = A(r)\cos\frac{\omega r}{c} + B(r)\sin\frac{\omega r}{c}. \tag{57}$$

Now, we are going to substitute (57) in (55) and we obtain the next two equations

$$\frac{d^2A}{dr^2} + \frac{\lambda}{r^2}A + \frac{2\omega}{c}\frac{dB}{dr} = 0 \tag{58}$$

And

$$\frac{d^2B}{dr^2} + \frac{\lambda}{r^2}B - \frac{2\omega}{c}\frac{dA}{dr} = 0, \tag{59}$$

Considering the fact that the coefficients of sine and cosine must be zero separately, due these functions has the same argument.

To solve (58) and (59) we propose $A(r) = ar^m$ and$B(r) = br^n$, where the coefficients a and b are constants and n, $m \in N$ are constants, too. Then, we obtain the following characteristic equations, substituting the solutions proposed in Eqs. (58) and (59)

$$am(m-1)+\lambda a+\frac{2\omega}{c}bnr^{n-m+1}=0 \qquad (60)$$

and

$$bn(n-1)+\lambda b-\frac{2\omega}{c}amr^{m-n+1}=0. \qquad (61)$$

Both equations can be satisfied for the next two cases:

Case1) $m=0, n=-1, \lambda=-2$ and $a=-\omega b/c$, and regarding Eq. (57) for $b=1$, we obtain

$$R=\frac{1}{r^2}\left(-\frac{\omega r}{c}\cos\frac{\omega r}{c}+\sin\frac{\omega r}{c}\right); \qquad (62)$$

Case2) $m=-1, n=0, \lambda=-2$ and $b=-\omega a/c$, and regarding Eq. (57) for $a=1$, we obtain,

$$R=\frac{1}{r^2}\left(\cos\frac{\omega r}{c}+\frac{\omega r}{c}\sin\frac{\omega r}{c}\right). \qquad (63)$$

Then, from the solutions (62) and (63) we have the general solution for Eq. (55)

$$R(r)=\frac{C_1}{r^2}\left(-\frac{\omega r}{c}\cos\frac{\omega r}{c}+\sin\frac{\omega r}{c}\right)+\frac{C_2}{r^2}\left(\cos\frac{\omega r}{c}+\frac{\omega r}{c}\sin\frac{\omega r}{c}\right), \qquad (64)$$

Where C_1 and C_2 are arbitrary constants. This general solution can be expressed as

$$R(r)=\frac{C}{r^2}\left\{\cos\left(\frac{\omega r}{c}-\alpha\right)+\sin\left(\frac{\omega r}{c}-\alpha\right)\right\}, \qquad (65)$$

Where C and α are arbitrary constants.

Now, we considering the Eq. (56), which for $\lambda=-2$ become:

$$\frac{d}{d\theta}\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}(\Theta\sin\theta)\right\}+2\Theta=0. \qquad (66)$$

This equation has the general solution:

$$\Theta(\theta)=C_3\sin\theta+C_4(\cot\theta-\sin\theta\ln|\csc\theta-\cot\theta|) \qquad (67)$$

Where C_3 and C_4 are arbitrary constants too.

Due to in $\theta=(2n+1)\pi$ the corresponding solution has a singularity, we can make $C_4=0$. Also, due to the homogeneity of the equation for the vector $\boldsymbol{\sigma}$, we can make $C_3=1$.

In this way, we can write the solution of the Eq. (54) as follows

$$\sigma_\phi(r,\theta)=\frac{1}{r^2}\left\{\cos\left(\frac{\omega r}{c}-\alpha\right)+\frac{\omega r}{c}\sin\left(\frac{\omega r}{c}-\alpha\right)\right\}\sin\theta. \qquad (68)$$

We can use the system (51)-(53) to find $\sigma_r(r,\theta)$ and $\sigma_\theta(r,\theta)$, namely

$$\sigma_r(r,\theta)=\frac{2c}{\omega r^3}\left\{\cos\left(\frac{\omega r}{c}-\alpha\right)+\frac{\omega r}{c}\sin\left(\frac{\omega r}{c}-\alpha\right)\right\}\cos\theta \qquad (69)$$

and

$$\sigma_\theta(r,\theta)=\frac{c}{\omega r^3}\left\{\cos\left(\frac{\omega r}{c}-\alpha\right)+\frac{\omega r}{c}\sin\left(\frac{\omega r}{c}-\alpha\right)-\frac{\omega^2r^2}{c^2}\cos\left(\frac{\omega r}{c}-\alpha\right)\right\}\sin\theta. \qquad (70)$$

In order to write the solutions in a short way, we define

$$\zeta=\cos\left(\frac{\omega r}{c}-\alpha\right)+\frac{\omega r}{c}\sin\left(\frac{\omega r}{c}-\alpha\right)$$

and

$$\eta=\zeta-\frac{\omega^2r^2}{c^2}\cos\left(\frac{\omega r}{c}-\alpha\right),$$

such as we can write the solution of Eq. (44) in spherical coordinates as

$$\boldsymbol{\sigma}=\xi\left(\frac{2\zeta}{r^3}\cos\theta\right)\hat{\boldsymbol{r}}+\xi\left(\frac{\eta}{r^3}\sin\theta\right)\hat{\boldsymbol{\theta}}+\xi\left(\frac{\omega\zeta}{cr^2}\sin\theta\right)\hat{\boldsymbol{\varphi}}. \qquad (71)$$

Where we have multiplied by $\xi\omega/c$ for convenience, and ξ has dimensions $[\xi]=g\ cm$.

In this way, we can see that the component in $\hat{\boldsymbol{\varphi}}$ direction corresponds to the vector $\boldsymbol{\gamma}$ and the components in $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ directions correspond to the vector $\boldsymbol{\kappa}$. That is,

$$\boldsymbol{\gamma}=\xi\left(\frac{\omega\zeta}{cr^2}\sin\theta\right)\hat{\boldsymbol{\varphi}} \qquad (72)$$

and

$$\boldsymbol{\kappa}=\xi\left(\frac{2\zeta}{r^3}\cos\theta\right)\hat{\boldsymbol{r}}+\xi\left(\frac{\eta}{r^3}\sin\theta\right)\hat{\boldsymbol{\theta}}. \qquad (73)$$

Finally, we write the solution for the gravitational and co gravitational fields as follows

$$\mathbf{G}=\xi\left\{\left(\frac{\omega\zeta}{cr^2}\sin\theta\right)\hat{\boldsymbol{\varphi}}\right\}\sin(\omega t-\delta) \qquad (74)$$

and

$$\mathbf{K}=\left\{\xi\left(\frac{2\zeta}{r^3}\cos\theta\right)\hat{\boldsymbol{r}}+\xi\left(\frac{\eta}{r^3}\sin\theta\right)\hat{\boldsymbol{\theta}}\right\}\cos(\omega t-\delta), \qquad (75)$$

where we have bear in mind temporal solutions (39), (40) and the spatial solutions (72), (73).

The necessary condition so that solutions (74) and (75) do not diverge in $r=0$ is,

$$\zeta(0)=\left\{\cos\left(\frac{\omega r}{c}-\alpha\right)+\frac{\omega r}{c}\sin\left(\frac{\omega r}{c}-\alpha\right)\right\}\Big|_{r=0}=0,$$

to fulfill such condition we need $\cos\alpha=0$, this implies $\alpha=(n+1/2)\pi$, where $n=0,\pm1,\pm2,\ldots$

Now, we calculate the next limits to ensure that the solutions converge,

$$\lim_{r\to 0}\frac{\zeta}{r^2}=0;\quad \lim_{r\to 0}\frac{\zeta}{r^3}=\frac{\omega^3}{3c^3}\text{ and }\quad \lim_{r\to 0}\frac{\eta}{r^3}=-\frac{2\omega^3}{3c^3},$$

these limits are evaluated for $\alpha=\pi/2$, also ζ and η were expanded in power series of r.

we have the energy density defined[5] as

$$w=-\frac{1}{8\pi}(G^2+K^2), \tag{76}$$

so, we obtain the limits

$$\lim_{r\to 0}\mathbf{G}=0;\quad \lim_{r\to 0}\mathbf{K}=\frac{2\xi\omega^3}{3c^3}\cos\omega t\,\hat{\boldsymbol{z}};\quad \lim_{r\to 0}w=\frac{\xi^2\omega^6}{18\pi c^6}\cos^2\omega t, \tag{77}$$

$\hat{\boldsymbol{z}}$ is the unit vector in the direction $Z+$ of the Cartesian system.

We propose $\alpha=\pi/2$ and $\delta=0$, because δ defines the initial wave phase of the fields **G** and **K**, we can write convergent solutions for these fields:

$$\mathbf{G}=\left\{\xi\left(\frac{\omega\zeta}{cr^2}\sin\theta\right)\hat{\boldsymbol{\varphi}}\right\}\sin\omega t \tag{78}$$

and

$$\mathbf{K}=\left\{\xi\left(\frac{2\zeta}{r^3}\cos\theta\right)\hat{\boldsymbol{r}}+\xi\left(\frac{\eta}{r^3}\sin\theta\right)\hat{\boldsymbol{\theta}}\right\}\cos\omega t, \tag{79}$$

where

$\zeta=-\frac{\omega r}{c}\cos\left(\frac{\omega r}{c}\right)+\sin\left(\frac{\omega r}{c}\right)$ and $\eta=\zeta-\frac{\omega^2r^2}{c^2}\sin\left(\frac{\omega r}{c}\right)$.

We can conclude that the solutions (78) and (79) for the Jefimenko's gravitational equations for the free space involve the novel existence of spherical formations of the gravitodynamical field.

Analysis of the Energy and the Energy Flow of the Gravitodynamical Field

The expression of energy density given by Eq. (76) can be changed after some algebraic manipulations in another that contains a part time-dependent and an independent one, namely:

$$w=-\frac{\xi^2}{16\pi}\left\{\frac{\omega^2\zeta^2}{c^2r^4}\sin^2\theta+\left[\frac{4\zeta^2}{r^6}\cos^2\theta+\frac{\eta^2}{r^6}\sin^2\theta\right]\right\}-\frac{\xi^2}{16\pi}\left\{\left[\frac{4\zeta^2}{r^6}\cos^2\theta+\frac{\eta^2}{r^6}\sin^2\theta\right]-\frac{\omega^2\zeta^2}{c^2r^4}\sin^2\theta\right\}\cos(2\omega t). \tag{80}$$

From this expression we find the geometric places where the energy density do not depend on time t. Those geometric places are

The points along the Z axis where is satisfied

$$\tan\left(\frac{\omega z}{c}\right)=\frac{\omega z}{c},$$

where $\theta=0,\pi$ and $\zeta=0$.

The surfaces where r satisfies

$$\eta^2=\zeta^2\left(\frac{\omega^2r^2}{c^2}-4\cot^2\theta\right).$$

The cross-section of such surfaces is drawn as discontinuous curves in the Fig. 3.

Now, we are going to obtain the gravitodynamical energy E_G inside a sphere of radius R centered at the origin, by means of

$$E_{TG}=\int_0^R\int_0^\pi\int_0^{2\pi}w(r,\theta,\phi,t)r^2\sin\theta\,drd\theta d\phi=E_G(R)+E_G(R,t), \tag{81}$$

where

$$E_G(R)=-\frac{\xi^2}{6R^3}\left[\frac{\omega^4R^4}{c^4}-\frac{\omega^2R^2}{c^2}\sin^2\left(\frac{\omega R}{c}\right)-\zeta^2\right], \tag{82}$$

and

$$E_G(R,t)=\frac{\xi^2}{6R^3}\zeta\eta\cos 2\omega t. \tag{83}$$

In this case,

$\zeta=-\frac{\omega R}{c}\cos\left(\frac{\omega R}{c}\right)+\sin\left(\frac{\omega R}{c}\right)$ And $\eta=\zeta-\frac{\omega^2R^2}{c^2}\sin\left(\frac{\omega R}{c}\right)$.

We can see from Eq. (83) that gravitodynamical energy does not change in time within spheres of radiuses R which are solutions of the next equations obtained respectively making $\zeta=0$ and $\eta=0$

$$\tan\left(\frac{\omega R}{c}\right)=\frac{\omega R}{c} \tag{84}$$

and

$$\tan\left(\frac{\omega R}{c}\right)=\frac{\frac{\omega R}{c}}{1-\frac{\omega^2R^2}{c^2}}. \tag{85}$$

The surfaces whose radii satisfy Eq. (84) contains only co gravitational field, there is not gravitational field over those surfaces, as we can verify from Eqs. (78) And (79) taking $\zeta=0$.

Now, we analyze the energy flow contained in the wave field given by (78) and (79).

As a first step we are going to calculate the Poynting's vector in GGSunits

[5]See, page 303 of Gravitation and Co gravitation [3] by Jefimenko. But in our case we use GGS units.

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{K}\times\mathbf{G} = \frac{\xi^2}{8\pi}\left\{\frac{\omega\zeta\eta\sin^2\theta}{r^5}\hat{\boldsymbol{r}} - \frac{\omega\zeta^2\sin(2\theta)}{r^5}\hat{\boldsymbol{\theta}}\right\}\sin(2\omega t). \quad (86)$$

Now, we will calculate the total momentum of the gravitodynamical field within a sphere of radius r centered at the origin. We will do this making use of the fact that the Poynting vector is proportional to the vector of the density of momentum, so that we can calculate the integral of the Poynting vector over the volume of the sphere of radius r. To make this easier we will express the unit vectors in spherical coordinates system in a Cartesian one, namely

$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}$ And $\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}}$

Integrating Eq. (81) over the given volume we obtain:

$$\iiint \mathbf{S}r^2\sin\theta\, dr d\theta d\phi = \frac{\xi^2\omega\sin(2\omega t)}{8}\int \frac{\zeta^2}{r^3}\sin^4\theta\Big|_0^{\pi} dr\hat{\boldsymbol{z}} = 0. \quad (87)$$

We can interpret this result as follows. The total momentum of the gravitodynamical field (78)-(79) in a volume bounded by an arbitrary sphere centered at the origin is *null* at any time given.

We will obtain the geometric places where the Poynting vector is zero at any instant. To do this, we need the conditions when the Poynting vector is zero and this is obtained by means of Eq. (76),

$$\zeta^2\sin 2\theta = 0 \;\text{ and }\; \zeta\eta\sin^2\theta = 0. \quad (88)$$

From the first equation of (88) we have

$\zeta = 0$, which satisfies both equations (88). We obtain the equation

$$\tan\left(\frac{\omega r}{c}\right) = \frac{\omega r}{c}. \quad (89)$$

Accordingly, the geometric places for the case (1) are spheres whose radiuses satisfy Eq. (89).

$\sin 2\theta = 0$. Which means that θ can be 0, $\pi/2$ or π.

- $\theta = 0, \pi$, In this case both equations satisfy the conditions (88). Therefore, the *geometric place* is the Z axis.
- $\theta = \pi/2$, we have two possibilities to satisfy the conditions (88), namely: $\zeta = 0$, as in case (1) or $\eta = 0$. From this condition we have

$$\tan\left(\frac{\omega r}{c}\right) = \frac{\frac{\omega r}{c}}{1 - \frac{\omega^2 r^2}{c^2}}. \quad (90)$$

So, we have the *geometric places* are rings in the plane $z = 0$ whose radiuses satisfying Eq. (90), these rings corresponding to the case $\theta = \pi/2$ and $\eta = 0$. In all points over these rings the cogravitational field is *zero.*

The Poynting vector is tangential in all points over these surfaces. This can be seen from Eq. (81). This fact clarifies the conservation of energy within spheres of radiuses (85).

The geometric places where the Poynting vector for the gravitodynamical field given by (78)-(79) is null at any time are:

- Z axis, called *cogravitational axis* because the ordinary gravitational field does not exist there.
- Rings at the plane $z = 0$ whose radiuses satisfy Eq. (90), called *gravitational rings* because there is not cogravitational field on them.
- Spheres centered at the origin whose radiuses satisfy Eq. (89), called *cogravitational spheres*, because there is not gravitational field on them.

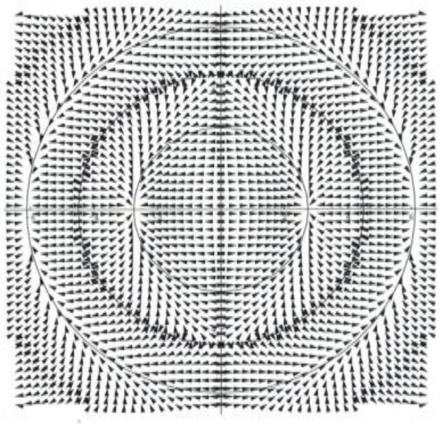

Figure1. *Poynting vector field distribution of the gravitodynamical field at a given time in the plane $x = 0$.*[6]

Let's see the graph where is shown the distribution of the Poynting vector field in order to clarify the results obtained in this section. Due to the axial symmetry of the energy density

[6]The graphics were performed in Mathematica™. The z axis is the ordinate and the y axis is the abscissa, and we put in the program $c = 1$ and $\omega = 1$ for simplicity.

and the energy-flux density, we can consider only the distribution in the plane $x = 0$.

In Fig. 1 we can see the vertical *cogravitational axis* that matches with the z axis, And we ca also see the cross-section of three spheres, which we will call *G*-sphere to the first one, *K*-sphere to the second one, and *G*-sphere to the last one[7], in an arbitrary instant. The total gravitodynamical energy conserves within *G*-spheres due to theenergy-flux vector at the surface of this sphere has tangential component only.

We can see also that energy transfers from these *G*-spheres to the gravitational ring (the equator of such spheres) and after a period defined by the function $\sin(2\omega t)$ in Eq. (81) the movement is reversed. Inside the first *G*-sphere the energy transfers from the *cogravitational axis* to the *gravitational ring*and having spent some time returns. The energy within the *K*-sphere is also conserved, we can see this because the Poynting vector is zero in every point of the *K*-sphere graphed. The energy is transferred from the surface of the *K*-sphere to the gravitational rings of the *G*-spheres. An analogue exchange of energy occurs between next *G*-spheres and *K*-spheres.

We want to emphasize the fact that the Poynting vector field reverses their direction after a time due to the function $\sin(2\omega t)$ present in Eq. (86).

Let us see the cross-section of the Poynting vector field in the plane $z = 0$ in Fig. 2.

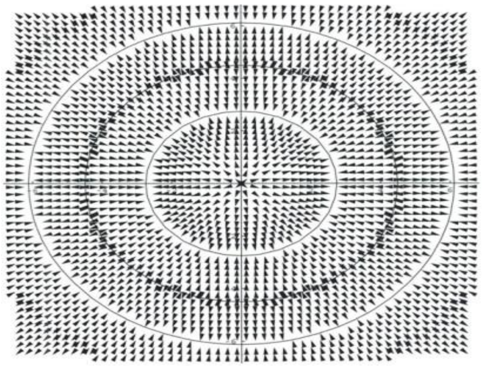

Figure2. *Pointing vector field distribution in the plane* $\mathbf{z} = \mathbf{0}$*, for a given instant of time*[8]

And at last, we can see the graphic of the cross-section in the plane $x = 0$ of the surfaces where the energy density is constant, the graphic of the first *K*-sphere; the graphic of the second *G*-spheres.

We need to note that these surfaces do not change in time in vacuum this means they do not deform nor displace when the time goes by.

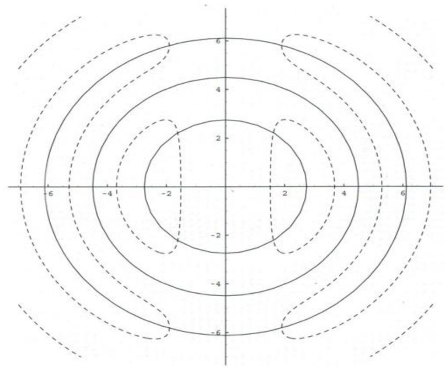

Figure3. *Cross-section of the surfaces where the energy density is constant (dashed lines). The continuous lines represent the K-sphere and the G-spheres.*

In Fig. 4 we obtain decreasing energy in the interval$R \in [0, 7]$, where we have plotted the graphics of the total gravitodynamical energy E_{TG} for four different time values, namely, $t = 0$, $\pi/4$, $\pi/2$ and $3\pi/4$. We have chosen these values due to the periodicity of the function $\cos 2\omega t$ in the time-dependent term of this energy. Here, we can see how the energy is changing in different the time values given before. There are various points where the gravitodynamical energy is constant for different values of time. For example, we can see that in the point (2.75, 3.8) all the curves intersect, this means, at $R = 2.74\ cm$ we obtain the total gravitodynamical energy $E_{TG} = -3.8 \times 10^{-3} erg$[9].

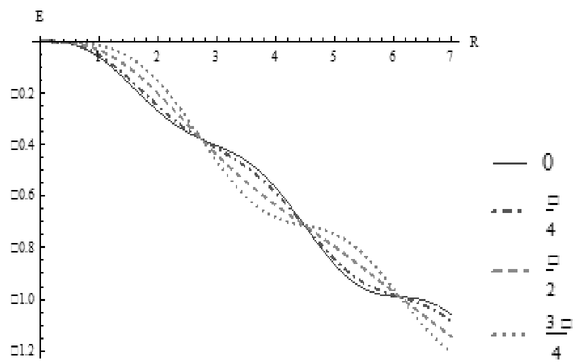

Figure4. *This graph shows how the total gravitodynamical energy* E_{TG} *alternates as the distance varies for the different values of time time* $t = 0, \pi/4, \pi/2$ *and* $3\pi/4$ *and for* $R \in [0, 7]$.

---

[7] Called *G*-spheres because they have a gravitational ring in the equator, and *K*-spheres because they are spheres of co gravitational field.

[8] *X*axis is the ordinate and *Y* axis is the abscissa.

[9] These values of distance and energy are only for reference, because we have to remind that we have chosen the values $\omega = 1$ and $c = 1$.

Now, we want to show both dependences in 3D graphics, and we are going to analyze them. Due to the periodicity of the term time-dependent we will fix them for $t \in [0, 2\pi]$. First, we have Fig. 5(a) the interval $R \in [0, 1]$.

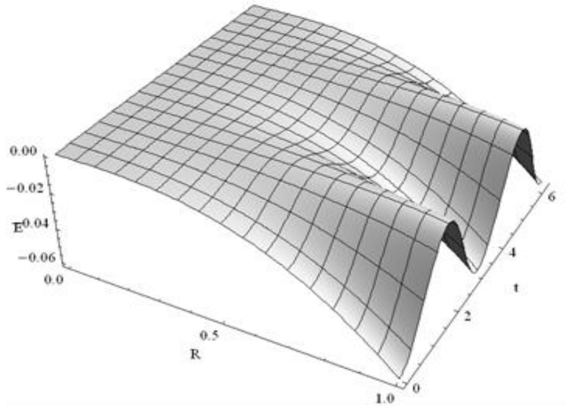

Figure5(a). *Graph of the total gravitodynamical energy contained in the G- and K-spheres.* E_{TG} *for the intervals* $R \in [0, 1]$ *and* $t \in [0, 2\pi]$.

In the Fig. 5 (b) we show the total gravitodynamical energy in the interval $R \in [0, 10]$.

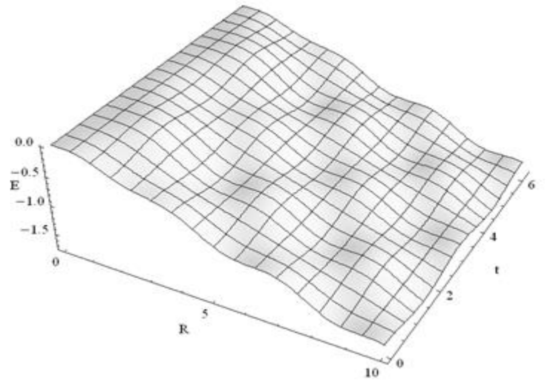

Figure5(b). *Total gravitodynamical energy in the interval* $R \in [0, 10]$.

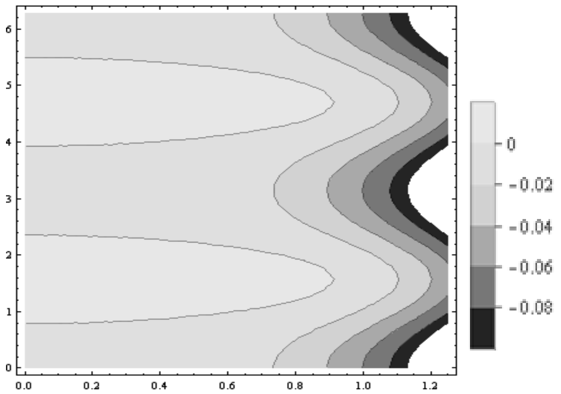

Figure6. *Contour plot of the total gravitational energy in the intervals* $R \in [0, 1.25]$ *and* $t \in [0, 2\pi]$. *The cross-section where the total gravitodynamical energy is null, forms semi-ovoid.*

At last, we want to show the graphics of the cross-sections of both fields, in Fig. 7 we have the gravitational field in the plane $z = 0$. In Fig. 8 we have drawn cross-sections of the cogravitational field in the planes $y = 0$ and $x = 0$ respectively.

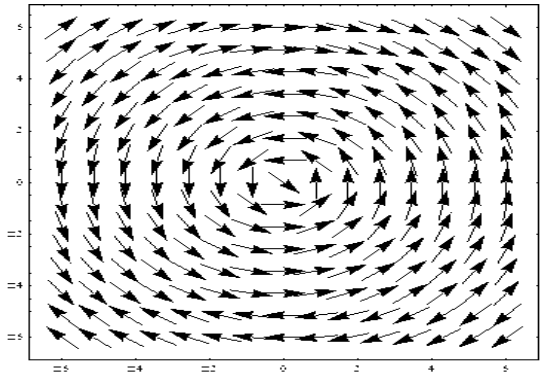

Figure7. *Ring-like form of the gravitational field in the plane* $z = 0$.

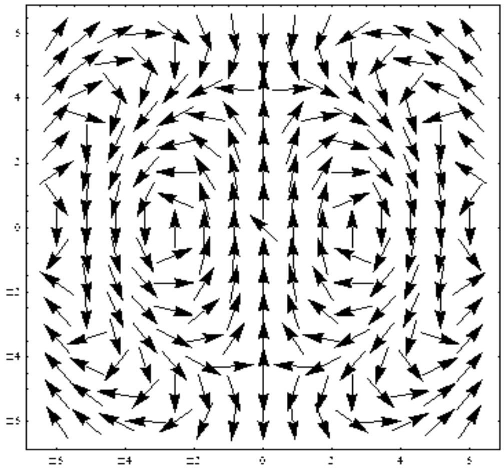

Figure8. *Cogravitational field in the planes* $y = 0$ *and* $x = 0$, *this field does not have components in the plane* $z = 0$.

Convergent Solution (78) and (79) Represented as a Superposition of Two Divergent Solutions

We can represent solutions (78) and (79) as a superposition of two waves spreading in opposite directions in each point, as the same way as Eq. (21). To do that only we need to do an algebraic transformation.

We call

$$\mathbf{G}_c = \mathbf{G}_{(\rightarrow)} + \mathbf{G}_{(\leftarrow)} \tag{91}$$

Gravitational convergent solution. This $\mathbf{G}_c$ is the superposition of the two waves $\mathbf{G}_{(\rightarrow)}$ and $\mathbf{G}_{(\leftarrow)}$ spreading in opposite directions at every point. In a similar way, we call

$$\mathbf{K}_c = \mathbf{K}_{(\rightarrow)} + \mathbf{K}_{(\leftarrow)} \tag{92}$$

co gravitational convergent solution. These solutions converge in $r = 0 \Leftrightarrow \delta = (n + 1/2)\pi$, where, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

Where in both cases we have

$$\mathbf{G}_{(\rightarrow)} = \frac{\xi\omega\sin\theta}{2cr^2}\left[\cos\left(\frac{\omega r}{c} - \omega t\right) + \frac{\omega r}{c}\sin\left(\frac{\omega r}{c} - \omega t\right)\right]\widehat{\boldsymbol{\phi}}, \tag{93}$$

$$\mathbf{G}_{(\leftarrow)} = -\frac{\xi\omega\sin\theta}{2cr^2}\left[\cos\left(\frac{\omega r}{c} + \omega t\right) + \frac{\omega r}{c}\sin\left(\frac{\omega r}{c} + \omega t\right)\right]\widehat{\boldsymbol{\phi}}, \tag{94}$$

$$\mathbf{K}_{(\rightarrow)} = \frac{\xi\cos\theta}{r^3}\left[\sin\left(\frac{\omega r}{c} - \omega t\right) - \frac{\omega r}{c}\cos\left(\frac{\omega r}{c} - \omega t\right)\right]\hat{\boldsymbol{r}}$$

$$+\frac{\xi\sin\theta}{2r^3}\left[-\frac{\omega r}{c}\cos\left(\frac{\omega r}{c} - \omega t\right) + \left(1 - \frac{\omega^2 r^2}{c^2}\right)\sin\left(\frac{\omega r}{c} - \omega t\right)\right]\widehat{\boldsymbol{\theta}}, \tag{95}$$

$$\mathbf{K}_{(\leftarrow)} = \frac{\xi\cos\theta}{r^3}\left[\sin\left(\frac{\omega r}{c} + \omega t\right) - \frac{\omega r}{c}\cos\left(\frac{\omega r}{c} + \omega t\right)\right]\hat{\boldsymbol{r}}$$

$$+\frac{\xi\sin\theta}{2r^3}\left[-\frac{\omega r}{c}\cos\left(\frac{\omega r}{c} + \omega t\right) + \left(1 - \frac{\omega^2 r^2}{c^2}\right)\sin\left(\frac{\omega r}{c} + \omega t\right)\right]\widehat{\boldsymbol{\theta}}. \tag{96}$$

The solutions (93)-(96) are solutions of the free Jefimenko's equations.

It is also possible demonstrate the next equations

$$\mathbf{G}_{(\rightarrow)} = \tfrac{1}{2}(-\mathbf{G}_d + \mathbf{G}_c) \text{ and } \mathbf{G}_{(\leftarrow)} = \tfrac{1}{2}(\mathbf{G}_d + \mathbf{G}_c) \tag{97}$$

and

$$\mathbf{K}_{(\rightarrow)} = \tfrac{1}{2}(-\mathbf{K}_d + \mathbf{K}_c) \text{ and } \mathbf{K}_{(\leftarrow)} = \tfrac{1}{2}(\mathbf{K}_d + \mathbf{K}_c), \tag{98}$$

where $\mathbf{G}_d$ and $\mathbf{K}_d$ are divergent solutions of the system (11)-(14):

$$\mathbf{G}_d = \left\{\xi\left(\frac{\omega\zeta_d}{cr^2}\sin\theta\right)\widehat{\boldsymbol{\phi}}\right\}\sin\omega t \tag{99}$$

and

$$\mathbf{K}_d = \left\{\xi\left(\frac{2\zeta_d}{r^3}\cos\theta\right)\hat{\boldsymbol{r}} + \xi\left(\frac{\eta_d}{r^3}\sin\theta\right)\widehat{\boldsymbol{\theta}}\right\}\cos\omega t, \tag{100}$$

where

$$\zeta_d = \cos\left(\frac{\omega r}{c}\right) + \frac{\omega r}{c}\sin\left(\frac{\omega r}{c}\right) \text{ and } \eta_d = \zeta_d - \frac{\omega^2 r^2}{c^2}\sin\left(\frac{\omega r}{c}\right).$$

We said before that $\mathbf{G}_{(\rightarrow)}$,$\mathbf{G}_{(\leftarrow)}$, $\mathbf{K}_{(\rightarrow)}$ and $\mathbf{K}_{(\leftarrow)}$ are solutions of the free Jefimenko's equations for gravitation and they are divergent in $r = 0$. And we can conclude this section emphasize the fact that this kind of gravitational waves of Jefimenko's solutions allows interference phenomenon as a superposition of two gravitational waves spreading in opposite directions.

CONCLUSIONS

As we have seen in section II, Eq. (21) represents a function of two waves spreading in opposite direction, both in $\boldsymbol{k}$ direction. Such waves are similar to electromagnetic waves, that is, $\mathbf{G}$ and $\mathbf{K}$ are transversal waves, perpendicular to the direction of propagation defined by the Pointing vector $\mathbf{S}$.

The superposition of the mentioned waves spreading one from the origin to infinity and the other one from the infinity to the origin produces stationary waves. Both waves have axial symmetry. So, we have obtained a *free* stationary gravitodynamical field, it is consequence of gravitational interference processes.

In this gravitodynamical configuration surfaces (the dashed lines in Fig. 3) and points (in the z axis) exist, where the energy density is *constant*. Such surfaces and points are nodes of *energy density* waves.

The cogravitational spheres obtained, can be containers of mass, because if we consider Eq. (4), any cogravitational field exerts a perpendicular force on any mass to the plane formed by the vectors $\mathbf{v}$ and $\boldsymbol{K}$, where $\boldsymbol{v}$ is the velocity of the particle and we have already seen that the cogravitational field in these unusual formations is tangent in every point of such surfaces. That is why the particles contained inside these spheres cannot leave such co gravitational spheres. The gravitational rings will take the particles of the gas contained in it and they will turn them in a direction of rotation of such a field.

From figure 4 we can see the points where the energy is constant, such points are those where the different curves are intersected and they represent the nodes of *energy* waves.

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