Dec. 2013, Volume 3, Issue 6, PP.162-164 Scientific Journal of Mathematics Research

The Simplest Way to Prove That Gauge Transformation between the Coulomb and Lorenz Gauges Must Not Exist

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Abstract- ***In this brief paper we propose to prove a theorem on non-classical symmetries for systems of differential equations whose solutions are related by means of a linear constraint. What we obtain is a representation of the invariance group of the system given by*** $(\cap_{i=1}^{N} G_i) \cup G_{Non}$ ***where the groups*** G_i ***are the classical Lie groups of each differential equation in the system taken separately, and*** G_{Non} ***is the non-classical symmetry group obtained when the set of determining equations is solved for the whole coupled system. Once we have the theorem, we apply it to the system of equations that define the scalar potential in the Coulomb and Lorenz gauges. It is well-known that the solutions of the scalar potential in the Coulomb gauge are related to the solutions of the scalar potential in the Lorenz gauge by a gauge transformation, however, these equations have been not considered in the literature from the point of view of its constrained symmetries. When this is done, it is possible to prove that the scalar potential in the Lorenz gauge cannot accept the Lorentz boost as a symmetry, which is excluded by the linear constraint introduced by the gauge transformation.***

Keywords- ***Electromagnetic Field; Lorenz and Coulomb Gauges***

I. INTRODUCTION

Let us describe the setting and nature of the problem we propose to deal with in this paper. Consider N linear differential operators F_i, i=1,.., N of orders n_i , i=1,..., N so that determine N functions φ_i: $\mathbf{R}^4 \to \mathbf{R}$ through the equations: $F_i(\varphi_i)=\rho_i$ where ρ_i: $\mathbf{R}^4 \to \mathbf{R}$ are given continuous functions.

The symmetry groups of the differential equations are G_i, i=1,...,N generated by vector fields X_{ij}, i=1,..,N, j= 1,...,N_i with N_i the number of symmetries accepted by the equation i. Each vector field generator satisfies, for each i and all j, the following equations:

$$X_{ij}^{n_i}[F_i(\varphi_i) - \rho_i] = 0 \tag{1}$$

$$F_i(\varphi_i) - \rho_i = 0 \tag{2}$$

where $X_{ij}^{n_i}$ means the n_i prolongation of the vector fields X_{ij} .When written out in coordinates these equations give us a system of linear overdetermined equations for the components of the vector fields X_{ij} which are known as classical Lie symmetries [1]. It is also well-known that classical Lie symmetries are not the only symmetries a set of differential equations accept as symmetries; P. Olver and E. Vorob´ev [3] have given many examples and a definition of non-classical symmetry and V. Fushchich and A. Nikitin [4] have generally explained why even non-classical symmetries are not the whole story. We propose to obtain the form of the symmetry group of system (1)-(2) when the simplest constraint equation among the variables is involved. Generally speaking we have a problem of non-classical symmetries when we try to solve system (1)-(2) plus a set of m differential constraints of the form: $f_k(\varphi_i, \nabla\varphi_i, \dots) = 0, k = 1, \dots, m$ where we use the symbol $\nabla\varphi_i$ as short notation for the full set of first order derivatives of the N functions φ_i. The problem we shall try to solve is now easily established: if we postulate a linear relationship among the functions φ_i of the form $\sum_{i=1}^{N} a_i\varphi_i = 0$ with $a_i \epsilon \mathbf{R}$ for all i, what is the form of the transformation group of this new symmetry problem with constraint? The answer of this question is proved in the next section.

II. THE THEOREM

We want to know the structure of the symmetry group of the following constrained system of differential equations:

$$F_i(\varphi_i) - \rho_i = 0 \tag{3}$$

$$\sum_{i=1}^{N} a_i\varphi_i = 0 \tag{4}$$

Instead of directly applying the prolongation of the vector fields we change (3)-(4) by the following equivalent set of differential equations i =1... N:

$$\sum_{k\neq i} a_k F_i(\varphi_k) - a_i\rho_i = 0, \tag{5}$$

where the sum is over all values of *k* except *i*. This change can be obtained using

$\varphi_i = -\frac{1}{a_i}\sum_k a_k\varphi_k$ where, again, the sum is over all values of *k* except *i*. In their paper, M. Aguero and R. Alvarado [5] has given an extended discussion of this sort of substitutions. Now the problem of symmetries is:

$$X^{n_i}[\sum_{k\neq i} a_k F_i(\varphi_k) - a_i\rho_i] = 0 \qquad i = 1, \dots, N \tag{6}$$

$$\sum_{k\neq i} a_k F_i(\varphi_k) - a_i\rho_i = 0 \qquad i = 1, \dots, N \tag{7}$$

System (6)–(7) is impressive; fortunately we do not have to solve it to achieve our goal.

Theorem. The symmetry group of the system (6)-(7) is of the form $(\cap_{i=1}^N G_i) \cup G_0$ where for all *i* we have $G_i \cap G_0 = \emptyset$.

Proof. If in the system (1)-(2) we use the constraint (4) we obtain the system (6)-(7), but with symmetries X_{ij} as solutions for each i, i.e. for a given *i* the symmetry X_{ij} is a symmetry of the equation *i* but not necessarily for all *i*. So we choose only those infinitesimal symmetries that generate each G_i which are symmetries of the differential equations (6)-(7) for all *i*. That is: we build the group $\cap_{i=1}^N G_{i_i}$. However, these are not the only solutions, because there could be non-Lie symmetries which can be obtained by solving the system (6)-(7) only. These symmetries generate the group G_0. Hence the symmetry group of (6)-(7) has the structure: $(\cap_{i=1}^N G_i) \cup G_0$ as claimed, and the symmetries contained in G_0 are not in any of the groups $G_{i,}$ otherwise, they would be redundant.

III. AN APPLICATION

Maxwell´s equations of classical electrodynamics can be solved introducing potentials [2] which are abelian gauge fields [6]. The gauge is fixed using conditions involving the potentials, like the Coulomb gauge, the Lorenz gauge or the several possible axial gauges. So when one wants the symmetries of the differential equations for the potentials, the gauge must be taken into account, and for just that reason the symmetries involved should be non-classical.

In the Lorenz gauge we can form a 4-vector function $A_\mu(x)$ that satisfies the equations: $A_\mu(x) = 4\pi j_\mu$ where $\square$ is the D´Alembert operator. Because the Lorentz gauge is relativistically covariant, the current j_μ a 4-vector and $\square$ an operator invariant under the Lorentz group, the whole system of Maxwell´s equations for the potentials in the Lorenz gauge is the same on all Lorentz frames.

However, this is not the case of the Coulomb´s gauge. It is well-known that the Coulomb´s gauge constraint is not relativistically invariant being valid only on a fixed reference system [4]. This can be proved directly making a Lorentz boost on the Coulomb´s gauge constraint, or over the Laplace equation for the scalar potential.

Because we believe that the correct description of nature must be relativistically covariant, the Coulomb gauge cannot be correct, but that using a gauge transformation to the Lorenz gauge this situation is changed. We assert, however, that such a hope is unwarranted.

Corollary. Let us suppose that for whatever reason we want a relationship of the form:

$\varphi_c = \varphi_L + \partial_t\gamma$ among a solution φ_c of the Poisson equation, φ_L of the D´Alembert equation and a function $\partial_t\gamma$ that is determined from differential equations arising from the postulated relationship and the Poisson and D´Alembert equations; in Ref. [7] Jackson calculates just one of this equations, and the pair equations are given in [8]. Hence this relationship defines a function that is not a solution of the D´Alembert equation under Lorentz boosts.

Proof. Let us call G_P and G_L the classical Lie groups of the Poisson and D´Alembert equations, respectively, which are not the same because the Coulomb´s gauge is not relativistically covariant like the Lorenz gauge, and the group of the function $\partial_t\gamma$ is, applying the previous theorem: $(G_P \cap G_L) \cup G_\gamma$ where G_γ is the non-classical symmetry group of the differential equations of the function $\partial_t\gamma$ not contained in any one of the groups G_P, G_L. So, the group of symmetries of the new problem is, again invoking the previous theorem:

$$(((G_P \cap G_L) \cup G_\gamma) \cap G_P \cap G_L) \cup G_{PL\gamma} = (G_P \cap G_L) \cup G_{PL\gamma}$$

The Lorentz boosts are not shared neither by G_P nor $G_{PL\gamma}$. This last group is the group of genuine non-Lie symmetries. Hence the relationship $\varphi_c = \varphi_L + \partial_t\gamma$ is not a solution of the D´Alembert equation when Lorentz boosts are applied upon it; i.e. if g is such a Lorentz boost, and we accept that $\varphi_L = \varphi_C - \partial_t\gamma$ is a solution to D´Alembert equation, then the transformed function $g^*\varphi_L$ is not a solution because of the existence of a gauge transformation.

IV. CONCLUSIONS

The corollary to the theorem of the previous section looks quite unacceptable, because it means that when we introduce a gauge transformation to change from a non-covariant gauge to a covariant gauge, what we obtain is that the covariant gauge is transformed into a non-covariant gauge, quite the opposite of our intentions.

If one looks at the literature, it is possible to find proofs that indeed quantum electrodynamics in any gauge, in Heisenberg operator formalism for quantum field theory, is covariant [9]. However, such a case is not related to the one focused in this paper because we are using c-fields not q-fields, so the transformations properties are not identical, but, more importantly, the Lorenz gauge constraint is valid all over the space for c-fields, but when q-fields are considered it can only be valid for a subset of the total Hilbert space to avoid contradictions with the commutation rules for the q-fields (see [10] Chapter 9). In this paper our results are related to c-fields and the proper way in which symmetries are restricted by the gauge transformation, a topic that has not been of the concern in the literature of classical electrodynamics, a subject that is full of surprises from the point of view of its symmetries yet, can be seen by reading *Symmetries of Maxwell´s Equations* [4].

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